HOMEOMORPHISMS PRESERVING A GOOD MEASURE IN A MANIFOLD.

by

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Thesis submitted for the degree of Doctor of Philosophy at the University of Warwick.

Mathematics Institute,
Warwick University,
July 1983.

Contents.

Acknowledgements.

My warmest thanks are to Prof. David Epstein. He encouraged me to develop the present dissertation; he told me how the mass flow homomorphism should be defined, and we worked out together the first stage of this project. His influence and friendship are some of the best things I will take to my country.

I am grateful to Rona and David Epstein for giving me a real home in Britain while I was completing this work.

To all the Maths. Institute: I would like to apologize if occasionally I was too worried to enjoy fully its pleasant atmosphere.

I want to thank Vicky Aguilar for her skilful typing of this thesis.

Finally, I would like to acknowledge the finantial support I received from the British Council, the Universidad Nacional de México and from beautiful CIMAT of Guanajuato during different periods of my research.

Declaration.

The main results in this thesis are original. They are generalizations of the work of other authors, especially that of von Neumann-Oxtoby-Ulam, M. Brown and A. Fathi. It is the work of A. Fathi which influenced the present research most deeply. Through A. Fathi, the author of this dissertation is indebted to many other mathematicians.

Theorem 1.15 is the product of joint research with my supervisor Prof. D.B.A. Epstein.

Summary.

Let M be a connected, finite dimensional, second countable manifold and let μ_0 be a locally finite, "∂-good" , positive Borel measure on M.

Let $H_c(M)$ be the group of all compactly supported homeomorphisms of M, and let $H_c(M, \mu_0)$ be the group of all measure preserving, compactly supported homeomorphisms of M. These groups are given the so called direct limit topology.

The purpose of these thesis is to prove the following results.

Theorem. The group $H_c(M, \mu_0)$ is locally contractible (see 4.9).

Theorem. The inclusion $H_c(M, \mu_0) \hookrightarrow H_c(M)$ is a weak homotopy equivalence (see 4.11).

Remark. Similar results hold for homeomorphisms fixing the boundary of M pointwise.

Theorem. Let M be a connected, second countable manifold without boundary and of dimension $n \geq 3$, and let μ_0 be a "∂-good" measure on M. Let $H_{c,0}(M, \mu_0)$ be the path component of the identity in $H_c(M, \mu_0)$. Then the abelianization of $H_{c,0}(M, \mu_0)$ is isomorphic to a quotient of the first real homology group $H_1(M, \mathbb{R})$ of M by some discrete subgroup Γ. The group Γ vanishes whenever M is non-compact. The commutator subgroup of $H_{c,0}(M, \mu_0)$ is simple and it is generated by all those elements in $H_{c,0}(M, \mu_0)$ which are supported in topological n-balls (see 8.14 and 8.16).

Introduction.

The group of homeomorphisms acts on the space of measures on a manifold, and the group of homeomorphisms preserving a given measure is just the stabilizer of that measure under the action. Broadly speaking, we will first look for a good orbit of measures. Then, we will compare the stabilizer of a measure on that orbit with the larger transformation group. Finally, we shall concentrate on some algebraic questions concerning the group of measure preserving homeomorphisms.

Let M^n be a connected, second countable manifold of dimension n. Let $\mathcal{H}(M)$ be the group of homeomorphisms of M, and let $\mathcal{M}(M)$ be the set of Radon measures on M; that is, the set of all positive, locally finite measures defined on the σ-algebra of Borel subsets of M. Then $\mathcal{H}(M)$ acts on $\mathcal{M}(M)$: if h is a homeomorphism, μ a Radon measure and B a Borel set on M, then the action of h on μ evaluated at B is $\mu(h^{-1}(B))$.

In [20], Oxtoby and Ulam characterized the orbit of standard Lebesgue measure on the unit cube I^n under this action. They did so by saying that the following simple list is a complete set of invariants:

(1) Total mass is equal to one;

(2) There are no points with positive measure;

(3) Non-empty open sets have positive measure;

(4) The boundary of the manifold has measure zero.

Notice that these conditions make sense if we substitute any manifold for I^n .

Much later, in [4] , M. Brown proved that any given topological compact connected manifold M^n can be obtained as an identification space of I^n by identifying certain points of ∂I^n to certain other points of ∂I^n . This implies that properties (1) to (4) above characterize an orbit of measures in $\mathcal{M}(M)$. Exactly the same reasoning that Oxtoby and Ulam used to extend their result to a large class of compact polyhedra is valid in this situation.

By definition, measures satisfying (2), (3) and (4) are called ∂-good measures.

Starting with these results, A.Fathi developed a program to study the group of homeomorphisms preserving a ∂-good probability on a compact manifold (see [9]).

The aim of this thesis is to generalize the work of A.Fathi to the σ-compact case, (see the summary preceding this introduction). As a by-product we polish his results on the abelianization of the group of homeomorphisms of a manifold M which are measure preserving and isotopic to the identity. Our "non-compact" methods allow us to replace "handles" by "neighbourhoods". Therefore, we do not depend, as A.Fathi does, on the existence of a handle-body decomposition of the manifold.‡

Our programme, which runs parallel to that of A.Fathi, is described in these paragraphs.

‡ A topological manifold of dimension $n \neq 4,5$ has a handle-body decomposition. The cases in which n=4 or n=5 are still unsettled (see Kirby and Siebenmann [15]).

We start with the problem of the equivalence of measures under homeomorphisms in a σ-compact manifold M. In [3], David Epstein and the author generalized the von Neumann-Oxtoby-Ulam theorem to the σ-compact case. When M is not compact, ∂-good probabilities are all equivalent under homeomorphisms, but the infinite ∂-good measures may split into different orbits. An example of this phenomenon is the fact that there is no homeomorphism of $\langle -\infty,\infty \rangle$ onto $\langle o,\infty \rangle$ preserving Lebesgue measure.

In [18], Moser proved the analogue of the von Neumann-Oxtoby-Ulam theorem for smooth C^∞ manifolds and volume forms. Its generalization to σ-compact smooth manifolds is due to Greene and Shiohama (see [10]).

Apparently, [3] is concerned only with the discrete problem of showing the existence of a homeomorphism which throws one of two fixed ∂-good measures into the other, whenever both of these measures have the same total mass and have the same "behaviour on ends". Therefore, in section 1 below, the continuity of the action $\mathcal{H}(M) \times \mathcal{M}(M) \to \mathcal{M}(M)$ is investigated. Also, we inspect more closely the well known concept of an "end" of a space.

Section 2 generalizes the above mentioned theorem of M.Brown to σ-compact manifolds.

Suppose that μ_p and ν_p are measures on a manifold M depending continuously on a parameter p. Suppose further that $\mu_p(M) = \nu_p(M)$ and that μ_p and ν_p agree on ends for each p. In section 3, we tackle the problem of whether we can find a homeomorphism h_p depending continuously on p and such that h_p sends μ_p into ν_p. We fail to do this. If we impose the additional condition that μ_{p_1} and μ_{p_2} have the same sets of measure zero for each p_1 and p_2, then we can find the desired continuous family of homeomorphisms $\{h_p\}_p$.

In sections 1 and 3, we work simultaneously with general homeomorphisms in the compact-open topology, and with compactly supported homeomorphisms in the direct limit topology.

In section 4, the results of Černvaskii, Edwards-Kirby on the deformation of spaces of embeddings (see [7]) are applied to show that the group $H_c(M, \mu_o)$ of measure preserving, compactly supported homeomorphisms of a manifold M is locally contractible in the direct limit topology.

By using the methods of Eilenberg and Wilder (see [8]), we finally prove, in section 4, that the group $H_c(M, \mu_o)$ is weakly homotopy equivalent to the group $H_c(M)$ of compactly supported homeomorphisms.

Section 5 is an important application of the Kirby-Edwards results and the von Neumann-Oxtoby-Ulam theorem.

Example: Let A be the 2-dimensional annulus bounded by the circle of radius two and the unit circle. Let C be a small "slice" at the top of A, and let U be a neighbourhood of C in A as shown in figure 1.

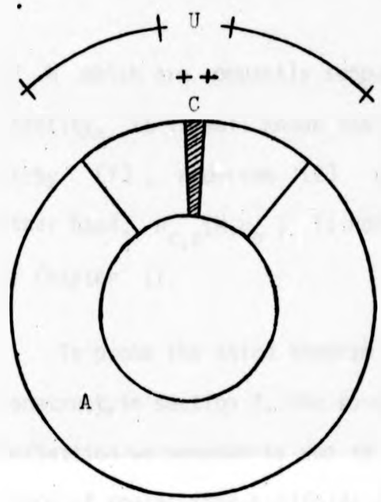


FIGURE 1

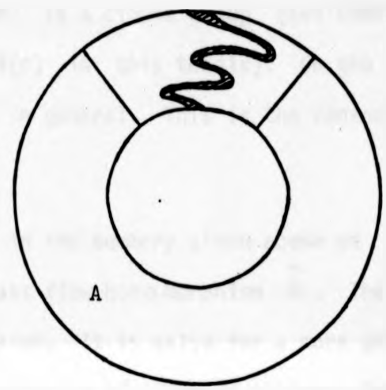


FIGURE 2

If $\iota : C \hookrightarrow A$ is a small "nice" measure preserving perturbation of C in $\mathring{U}$ as in figure 2, then we have the following two results: Firstly, the Kirby-Edwards theorem states that $\iota : C \hookrightarrow A$ can be extended to a homeomorphism $\bar{\iota}$ of A in such a way that $\bar{\iota}$ is the identity outside U. Secondly, one of the results in Section 5 says that $\iota : C \hookrightarrow A$ can be extended to a <u>measure preserving</u> homeomorphism of A, but the "wave" created by the perturbation of C most probably will have to propagate all around A.

This is a crucial difference of behaviour between general homeomorphisms and measure preserving homeomorphisms. Certainly this observation lies in the heart of the study of the mass flow homo morphism in Chapter II.

Let M be a connected manifold without boundary, and let μ_o be a good measure on M. Denote by $H_{c,o}(M)$ the group of homeomorphisms of M which are compactly supported and compactly isotopic to the identity. It is well known that $H_{c,o}(M)$ is a simple group (see Edwards-Kirby [7], Anderson [2] and 8.16(c) in this thesis). On the other hand, $H_{c,o}(M,\mu_o)$ is not simple in general. This is the content of Chapter II.

To prove the third theorem stated in the summary given above we construct, in section 7, the so called mass flow homomorphism $\tilde{\theta}$. The definition we propose is due to W.Thurston. It is valid for a more general class of spaces than manifolds and it makes use of a homology theory based on measures, due also to W.Thurston.

The mass flow homomorphism says how much mass flows around a particular cycle during an isotopy to the identity. Its domain is the universal covering space $\tilde{H}_{c,o}(M, \mu_o)$ of $H_{c,o}(M, \mu_o)$ and its range is the first homology vector space $H_1(M, \mathbb{R})$. These are discussed in section 6.

Clearly, the covering projection $\tilde{H}_{c,o}(M, \mu_o) \longrightarrow H_{c,o}(M, \mu_o)$ is a group homomorphism having as kernel the fundamental group Π of $H_{c,o}(M, \mu_o)$. Let Γ_M be the image of Π under $\tilde{\theta}$. Then it is not difficult to construct the following commutative diagram with exact columns.

$$\begin{tikzcd}
\operatorname{Ker}\tilde{\theta} \cap \Pi \arrow[r, hook] \arrow[d, hook] & \Pi \arrow[r, two heads] \arrow[d, hook] & \Gamma_M \arrow[d, hook] \\
\operatorname{Ker}\tilde{\theta} \arrow[r, hook] \arrow[d, two heads] & \tilde{H}_{c,o}(M, \mu_o) \arrow[r, "\tilde{\theta}"] \arrow[d, two heads] & H_1(M, \mathbb{R}) \arrow[d, two heads] \\
\operatorname{Ker}\theta \arrow[r, hook] & H_{c,o}(M, \mu_o) \arrow[r, "\theta"] & H_1(M, \mathbb{R})/\Gamma_M
\end{tikzcd}$$

where θ is the homomorphism induced in the quotient by $\tilde{\theta}$ and it may be called ambiguously the mass flow homomorphism.

In some way, this important diagram contains all the algebraic information about $H_{c,o}(M, \mu_o)$.

Section 7 is also devoted to the inspection of the above diagram. In particular, Γ_M is a discrete subgroup of $H_1(M, \mathbb{R})$ which is zero when M is non-compact and connected.

Suppose that the space A in Figure 1 of this introduction does not represent a 2-dimensional annulus, but a neighbourhood of some essential embedded circle γ in the manifold M. Then we can perturb the "slice" C in order to force a positive amount of mass to "circulate" around A. The conclusion is that this perturbation has non-zero mass flow along γ. Using this idea, we prove, in section 8, that $\tilde{\theta}$ is surjective.

It is then shown that the kernel Ker θ of the map $\theta: H_{c,o}(M, \mu_o) \to H_1(M, \mathbb{R})/\Gamma$ is generated by its elements supported in topological balls.

To obtain that Ker θ is simple and equal to the commutator subgroup of $H_{c,o}(M, \mu_o)$ it is necessary to assume that M is a connected manifold with empty boundary and of dimension $n \geq 3$.

If M is the unit circle S^1, then $H_{c,o}(M, \mu_o)$ is isomorphic to the group S^1 of rotations. If M is the real line $\mathbb{R}$, then $H_{c,o}(M, \mu_o)$ is trivial. In both cases, Ker θ is trivial and the above results hold as well. On the other hand, the case in which M is of dimension $n=2$ remains unsettled.

Chapter I. *The topology.*

§1. Homeomorphisms, measures and ends.

1.1 Definitions.

Let X be a locally compact, locally connected, Hausdorff space. If A is contained in X, its closure will be denoted by $\mathrm{Cl}\, A$. If $h: X \to X$ is a homeomorphism of X onto itself, the support of h is the set

$$\operatorname{supp} h = \mathrm{Cl}\, \{x \in X \mid h(x) \neq x\}.$$

We say that h has compact support if $\operatorname{supp} h$ is compact.

Denote by $H(X)$ the group of homeomorphisms of X and by $H_c(X)$ the group of homeomorphisms of X with compact support.

Recall that the compact-open topology κ on $H(X)$ is the topology having for a subbase the sets

$$[K, U] = \{ h \in H(X) \mid h(K) \subset U \},$$

for K compact and U open in X.

It can be proved that, with this topology, $H(X)$ becomes a topological group (see Siebenmann [22]).

If X is second countable, locally compact, locally connected and Hausdorff, then the compact-open topology κ can be defined in terms of a metric d on X, (every regular, second countable, Hausdorff space is metrizable). A base for κ is given by the sets

$$(K,\varepsilon)\, h_0 = \{h \in H(X) \mid d(h_0(x), h(x)) + d(h_0^{-1}(x), h^{-1}(x)) < \varepsilon \text{ for all } x \in K\},$$

for K compact in X, $\epsilon > 0$ and $h_0 \in H(X)$.

If X is compact, then the formula

$$d^*(h_0, h) = \underset{x \in X}{\text{supremum}}\ d(h_0(x), h(x)) + \underset{x \in X}{\text{supremum}}\ d(h_0^{-1}(x), h^{-1}(x)),$$

defines a metric d^* on $H(X)$ compatible with κ.

In general, the additional hypothesis that X is second countable implies that X is σ-compact; so let $\{K_i\}_{i \in \mathbb{N}}$ be a locally finite countable covering of X by compact sets and define

$$d^*(h_0, h) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\frac{d_i^*(h_0, h)}{1 + d_i^*(h_0, h)} \right),$$

where

$$d_i^*(h_0, h) = \underset{x \in K_i}{\text{supremum}}\ d(h_0(x), h(x)) + \underset{x \in K_i}{\text{supremum}}\ d(h_0^{-1}(x), h^{-1}(x)).$$

Then d^* is a metric on $H(X)$ compatible with κ. Furthermore, $(H(X), d^*)$ is complete (see Willard [26]).

Observe that if Y is an arbitrary compact Hausdorff space, then $H(Y)$ is a topological group in the compact-open topology, but this is no longer true for a general non-compact space if we drop the hypothesis that Y is locally connected. That is, there are locally compact, Hausdorff spaces Y for which the inverse map $h \mapsto h^{-1}$ in $H(Y)$ is not continuous. The space $\{2^i \mid i \in \mathbb{Z}\} \cup \{0\} \subset \mathbb{R}$ is an example.

Let X be a locally compact, locally connected, Hausdorff space. Now we want to impose a topology on the group $H_c(X)$ of compactly supported homeomorphisms of X much finer than the compact open topology and better adapted to our purposes. First we define, for each $K \subset X$ compact,

$$H(K, X) = \{ h \in H(X) \mid \operatorname{supp} h \subset K \},$$

and put on $H(K, X)$ the compact-open topology (i.e. the subspace topology from $H(X)_K$).
It follows that

$$\{ H(K, X) \mid K \subset X \text{ compact} \}$$

is a closed family of spaces on $H_c(X)$, so we can give it the coinduced (or direct limit) topology, say $\varinjlim$ (see Appendix One).
Obviously,

$$H_c(X)_{\varinjlim} \xrightarrow{\text{Id}} H_c(X)_K$$

is continuous.

With the $\varinjlim$ topology, $H_c(X)$ is not in general a topological group. Nevertheless, for every $K, L \subset X$ compact,

$$(1)\quad H(K, X) \times H(L, X) \to H(K \cup L, X)$$
$$(h, g) \mapsto h \circ g$$

and

$$(2)\quad \mathcal{H}(K, X) \to \mathcal{H}(K, X)$$
$$h \mapsto h^{-1}$$

are well defined and continuous functions.

Hence

$$(3)\quad [\, \mathcal{H}_c(X) \times \mathcal{H}_c(X) \,]_{\underrightarrow{\lim}} \to \mathcal{H}_c(X)_{\underrightarrow{\lim}}$$
$$(h, g) \mapsto h \circ g$$

and

$$(4)\quad \mathcal{H}_c(X)_{\underrightarrow{\lim}} \to \mathcal{H}_c(X)_{\underrightarrow{\lim}}$$
$$h \mapsto h^{-1}$$

are continuous, where $[\, \mathcal{H}_c(X) \times \mathcal{H}_c(X) \,]_{\underrightarrow{\lim}}$ denotes the cartesian product $\mathcal{H}_c(X) \times \mathcal{H}_c(X)$ with the coinduced topology defined by the coherent family

$$\{\, \mathcal{H}(K, X) \times \mathcal{H}(L, X) \mid K, L \subset X \text{ compact} \,\} .$$

Let M^n be an n-dimensional (second countable) manifold possibly with non-empty boundary ∂M. Define

$$(1)\quad \mathcal{H}^{\partial}(M) = \{\, h \in \mathcal{H}(M) \mid h(x) = x \ \ \forall x \in \partial M \,\} ;$$

$$(2)\quad \mathcal{H}^{\partial}(K, M) = \mathcal{H}^{\partial}(M) \cap \mathcal{H}(K, M)$$

for each $K \subset M$ compact;

(3) $H_c^{\partial}(M) = H^{\partial}(M) \cap H_c(M)$.

Now, $H^{\partial}(M)$ is closed in $H(M)_k$ and $H_c^{\partial}(M)$ is closed in $H_c(M)_{\underset{\rightarrow}{\lim}}$.

1.2 Definitions.

Let X be a locally compact, second countable Hausdorff space. A Radon measure μ on X is a locally finite positive measure defined on the σ-algebra of all Borel subsets. The support of μ, denoted by supp μ, is the complement of the maximum open subset of X in which μ has zero measure. We say that μ is a good measure if it has no atoms (an atom is a point of non-zero μ-measure) and if its support is the whole of X.

Denote by $M(X)$ (resp. $M_g(X)$) the set of all Radon measures on X (resp. the set of good Radon measures on X). Observe that if $\mu \in M(X)$ then μ is regular (See Rudin [21]).

Let $f: X \to \mathbb{R}$ be an arbitrary real valued function on X. The support of f is the set

$$\text{supp } f = \text{Cl}\{x \in X \mid f(x) \neq 0\}.$$

In what follows, when writing supp f, it will always be clear if we are referring to the support of a real valued function, the support of a homeomorphism, or the support of a measure.

Define the weak topology ω on $M(X)$ as the smallest or coarsest topology such that, for each $f: X \to \mathbb{R}$ continuous with compact support, the function

$$\mu \mapsto \int_X f \, d\mu \qquad \forall \ \mu \in M(X)$$

is continuous.

There is an action

$$H(X) \times M(X) \to M(X)$$

$$(h, \mu) \mapsto h_* \mu$$

where $h_* \mu$ is defined by

$$h_* \mu(E) = \mu(h^{-1}(E))$$

for each Borel set $E \subset X$.

1.3 Lemma. (See Rudin [21, Chapter 2]).

Let X be a locally compact, second countable, Hausdorff space. Let $K \subset X$ be compact and $A \subset X$ be open. The two functions from $M(X)$ into $\mathbb{R}$ defined by

$$(1) \qquad \mu \mapsto \mu(K)$$

and

$$(2) \qquad \mu \mapsto \mu(A)$$

are, respectively, upper semicontinuous and lower semicontinuous. That is, if $a > 0$ the sets

$$\{\mu \in M(X) \mid \mu(K) < a\}$$

and

$$\{\mu \in M(X) \mid \mu(A) > a\}$$

are open subsets of $M(X)_\omega$.

We omit the proof.

□

Example:

Let $X = [0, 1]$, $K = \{0\}$ and, for each $n = 1,2,\ldots,$ let $\mu_n = dt + \alpha_{1/n}$, where dt is Lebesgue measure on $[0, 1]$ and $\alpha_{1/n}$ is the atomic probability concentrated at $\{1/n\}$.
Clearly $\mu_n \to dt + \alpha_0$, but $0 = \mu_n(K) \not\to 1$. Hence (1) above is not continuous in this case.

1.4 Proposition.

Let X be a locally compact, locally connected, second countable, Hausdorff space. Then the map

$$H(X)_K \times M(X)_\omega \longrightarrow M(X)_\omega$$

is continuous, where $H(X)$ is given the compact open topology and $M(X)$ the weak topology (see 1.2).

Proof.- (c f. Fathi [9,Prop. 1.5, p.94]).

Let $f: X \to \mathbb{R}$ be a continuous function with compact support. We want to prove that the function

$$H(X) \times M(X) \longrightarrow \mathbb{R}$$

$$(h, \mu) \mapsto \int_X f\, d\, h_* \mu = \int_X f \circ h\, d\, \mu$$

is continuous. For this purpose, fix $(h_o, \mu_o) \in H(X) \times M(X)$ and let $\varepsilon > o$.

Let U be a neighbourhood of supp f with compact closure and choose $C > o$ with $\mu_o(h_o^{-1}(\mathrm{Cl}\ U)) < C$.

If d is a metric on X compatible with its topology, then $f\colon (X, d) \to \mathbb{R}$ is uniformly continuous, so we can choose $\delta > o$ such that, for every $x, y \in X$, $| f(x) - f(y) | < \varepsilon/2C$ whenever $d(x,y) < \delta$.

Clearly, the following sets are open:

$$(1) \quad A_1 = \{ h \in H(X) \mid h^{-1}(\mathrm{supp}\ f) \subset h_o^{-1}(U) \} .$$

Note that if $h \in A_1$ and $x \notin h_o^{-1}(U)$ then $h(x) \notin$ supp f, so $f \circ h(x) = o$,

$$(2) \quad A_2 = (h_o^{-1}(\mathrm{Cl}\ U), \delta)\, h_o \qquad \text{(see 1.1)} .$$

In particular, if $h \in A_2$ and $x \in h_o^{-1}(\mathrm{Cl}\ U)$ then $| f \circ h(x) - f \circ h_o(x) | < \varepsilon/2C$,

$$(3) \quad B_1 = \{ \mu \in M(X) \mid \left| \int f \circ h_o d\, \mu - \int f \circ h_o\, d\, \mu_o \right| < \varepsilon/2 \},$$

$$(4) \quad B_2 = \{ \mu \in M(X) \mid \mu(h_o^{-1}(\mathrm{Cl}\ U)) < C \} .$$

Therefore, if $A = A_1 \cap A_2$ and $B = B_1 \cap B_2$ then $A \times B$ is a neighbourhood of (h_0, μ_0) in $H(X) \times M(X)$ and if $(h, \mu) \in A \times B$, then

$$\left| \int f \, d\, h_* \mu - \int f \, d\, h_{0*} \mu_0 \right|$$

$$\leq \left| \int (f \circ h - f \circ h_0) d\mu \right| + \left| \int f \circ h_0 d\mu - \int f \circ h_0 \, d\, \mu_0 \right|$$

$$\leq \left[\operatorname*{supremum}_{x \in Cl(h_0^{-1} U)} \left| f \circ h(x) - f \circ h_0(x) \right| \right] \mu_0 \, (h_0^{-1}(U)) + \varepsilon/2$$

$$\leq \frac{\varepsilon}{2C} \, C + \varepsilon/2 = \varepsilon$$

□

1.5 Lemma.

Let X be a locally compact, second countable, Hausdorff space. Let $f: X \to \mathbb{R}$ be a bounded function with compact support such that its set of singularities

$$\overline{S_{(f)}} = Cl \{ x \in X \mid f \text{ is discontinuous at } x \}$$

has empty interior. Then the function

$$\mu \mapsto \int_X f \, d\, \mu \qquad \mu \in M(X)$$

is well defined and is continuous at each point on the set

$$\{ \mu \in M(X) \mid \mu(\overline{S_{(f)}}) = o \} .$$

In particular, if A is any closed subset of X, then the function

$$M(X) \xrightarrow{\text{res}} M(A)$$

$$\text{res}(\mu) = \mu|_A \qquad \mu \in M(X)$$

is continuous on the set

$$\{ \mu \in M(X) \mid \mu(\text{Fr } A) = 0 \}$$

where $\text{Fr } A$ denotes the topological frontier of A.

Proof.

Let $\mu_0 \in M(X)$ be such that $\mu_0(\overline{S_{(f)}}) = 0$ and let $\varepsilon > 0$. Let $C > 0$ be an upper bound for f and let U be an open neighbourhood of $\overline{S_{(f)}}$ with compact closure such that $\mu_0(\text{Cl } U) < \varepsilon / 3C$.

Define $V = X \setminus \overline{S_{(f)}}$ and choose continuous functions $\psi_U, \psi_V : X \longrightarrow [0, 1]$ such that

$$\psi_U + \psi_V = 1,$$

$$\text{supp } \psi_U \subset U,$$

$$\text{supp } \psi_V \subset V.$$

Define

$$B_1 = \{ \mu \in M(X) \mid \mu(\text{Cl } U) < \varepsilon/2C \},$$

$$B_2 = \{ \mu \in M(X) \mid \left| \int \psi_V f \, d\mu - \int \psi_V f \, d\mu_0 \right| < \varepsilon/3 \} .$$

Hence, by Lemma 1.3 and the fact that $\psi_V f$ is continuous with compact support, the set $B = B_1 \cap B_2$ is an open neighbourhood of μ_0 in $M(X)$. Therefore, if $\mu \in B$ then

$$\left| \int f \, d\mu - \int f \, d\mu_0 \right| < \left| \int \psi_U f \, d(\mu - \mu_0) \right| + \left| \int \psi_V f \, d(\mu - \mu_0) \right|$$

$$< C(\mu(U) + \mu_0(U)) + \varepsilon/3$$

$$< 2\varepsilon/3 + \varepsilon/3 = \varepsilon.$$

This completes the proof.

□

1.6 Definitions.

Let X be a locally compact, locally connected, second countable, Hausdorff space. Let $\{ X_\alpha \mid \alpha \in A \}$ be the family of connected components of X and let K be a compact subset in X. Two measures $\mu, \nu \in M_g(X)$ are said to be K-related if and only if

(1) $\mu(K \cap X_\alpha) = \nu(K \cap X_\alpha)$ all $\alpha \in A$;

(2) $\mu|_{X \setminus K} = \nu|_{X \setminus K}$.

Denote by $M_g(K, X, \mu)$ the set of all (good) measures K-related to μ .

We say that two measures $\mu, \nu \in M_g(X)$ are compactly related if and only if they are K-related for some $K \subset X$ compact. Denote by $M_{g,c}(X,\mu)$ the set of all (good) measures compactly related to μ.

It follows from Lemma 1.3 that if $\mu \in M_g(X)$, $K \subset X$ is compact, then $M_g(K, X, \mu)$ is closed in $M_g(X)$. Indeed, suppose μ_1, ν_1 are (good) measures in X which are not K-related. Then we can choose a compact $C \subset X$ such that $\mu_1(C) \neq \nu_1(C)$ and either $C = X_\alpha \cap K$ for some $\alpha \in A$ or $C \cap K = \emptyset$. Assume, without loss of generality, that $\mu_1(C) < \nu_1(C)$ and choose an open set $U \supset C$ with compact closure such that $\mu_1(\mathrm{Cl}\, U) < \nu_1(C)$ and either $U \subset X_\alpha$ or $(\mathrm{Cl}\, U) \cap K = \emptyset$. Then

$$\{\mu \mid \mu(\mathrm{Cl}\, U) < \nu_1(C)\} \times \{\nu \mid \nu_1(C) < \nu(U)\}$$

is an open set in $M_g(X) \times M_g(X)$ (see 1.3) such that its intersection with the relation " μ is K-related to ν" is empty. Hence the relation is closed and, in particular, the induced equivalence classes are closed in $M_g(X)$.

Therefore, the collection

$$\{M_g(K,X,\mu) \mid K \subset X \text{ compact}\}$$

is a coherent family on $M_{g,c}(X,\mu)$. We denote, once again, the resulting coinduced topology by $\varinjlim$.

Let M be a (second countable) manifold.

Define

(1) $\mathcal{M}^{\partial}(M) = \{ \mu \in \mathcal{M}(M) \mid \mu(\partial M) = 0 \}$;

(2) $\mathcal{M}_g^{\partial}(M) = \mathcal{M}^{\partial}(M) \cap \mathcal{M}_g(M)$;

(3) $\mathcal{M}_g^{\partial}(K, M, \mu) = \mathcal{M}_g(K, M, \mu) \cap \mathcal{M}^{\partial}(M)$;

for every $K \subset M$ compact and $\mu \in \mathcal{M}_g^{\partial}(M)$;

(4) $\mathcal{M}_{g,c}^{\partial}(M, \mu) = \mathcal{M}_{g,c}(M, \mu) \cap \mathcal{M}^{\partial}(M)$;

for each $\mu \in \mathcal{M}_g^{\partial}(M)$.

1.7 Proposition.

Let X be a connected, locally compact, locally connected, second countable, Hausdorff space. Let $\mu_o \in \mathcal{M}_g(X)$. Then

$$(1) \quad \begin{aligned} [\, \mathcal{H}_c(X) \times \mathcal{M}_{g,c}(X, \mu_o)]_{\underset{\to}{\lim}} &\to \mathcal{M}_{g,c}(X, \mu_o)_{\underset{\to}{\lim}} \\ (h, \mu) &\mapsto h_* \mu \end{aligned}$$

is continuous.

Proof.

By Proposition 1.4, the action

$$\mathcal{H}(X)_{\kappa} \times \mathcal{M}_g(X)_{\omega} \to \mathcal{M}_g(X)_{\omega}$$

is continuous. Now let K, L be compact subsets in X, let $h \in \mathcal{H}(K, X)$ and let $\mu \in \mathcal{M}_g(L, X, \mu_o)$. Then $h_* \mu \in \mathcal{M}_g(K \cup L, X, \mu_o)$ so (1) above is well defined and stratified (see Appendix One, statement (3)) when the coherent family of spaces

$$\{\, H(K, X)_\kappa \times M_g(L, X, \mu_0)_\omega \mid K, L \subset X \text{ compact} \,\}$$

is defined on $H_c(X) \times M_{g,c}(X, \mu_0)$.

□

1.8 Remark

Proposition 1.7 and its proof are typical in this work:

First we have a continuous function, say $f: X^1 \to X^2$.

Secondly, we observe that f can be restricted to $f_c: X_c^1 \to X_c^2$ where X_c^i $(i = 1,2)$ has been defined as the union of a certain collection K^i of closed sets in X^i.

Thirdly, we check that $f_c: X_c^1 \to X_c^2$ is stratified (with respect to the coherent families K^1 and K^2) and therefore $f_c: X_c^1 \to X_c^2$ is continuous when X_c^i $(i = 1,2)$ is given the topology coinduced by the family K^i.

Note that $M_g^\partial(M)$ is not closed in $M_g(M)_\omega$, although $H^\partial(M)$ is closed in $H(M)_\kappa$ (e.g. consider the sequence $\{\mu_n \times dt\}_{n=1}^\infty$ in $M_g^\partial(I \times I)$, where μ_n is given in the example at 1.3).
Hence we must prove the following corollary by using the above pattern, and not by simply restricting the action in Proposition 1.7, (c f. Appendix One, statement (2)).

1.9 Corollary.

If $X = M$ is a (second countable) manifold, and $\mu_0 \in M_g^\partial(M)$, then

$$[\, H_c^\partial(M) \times M_{g,c}^\partial(M, \mu_0) \,]_{\varinjlim} \to M_{g,c}^\partial(M, \mu_0)_{\varinjlim}$$

is continuous.

Proof.

The proof is the same as that in Proposition 1.7 if we substitute $\mathcal{H}^{\partial}(M)$ for $\mathcal{H}(X)$ and $\mathcal{M}_g^{\partial}(M)$ for $\mathcal{M}_g(X)$.

□

1.10 Definitions.

Let X be a locally compact, Hausdorff space, and let $K \subset X$ be compact. We denote by $C(X \setminus K)$ the set of connected components of $X \setminus K$ considered as a discrete topological space.

If K, L are compact subsets of X with $K \subset L$, then there is a well defined continuous function

$$\rho_K^{\,L} : \; C(X \setminus L) \to C(X \setminus K)$$

such that for each $V \in C(X \setminus L)$, $\rho_K^{\,L}(V)$ is the unique component of $X \setminus K$ containing V. Then the collection

$$\{ K, \rho_K^{\,L} \mid K, L \subset X \text{ compact and } K \subset L \}$$

constitutes an inverse system of topological spaces (see Dugundji [6 , Appendix Two]).

An end of X is, by definition, a point in the inverse limit space of this system. In other words, an end of X is a function e which assigns to each compact set K of X a non-empty connected component $e(K)$ of $X \setminus K$, in such a way that $K_1 \subset K_2$ implies $e(K_2) \subset e(K_1)$. Let $E(X)$ be the set of all ends and define a topology on $X \cup E(X)$ by defining, for each compact K, a typical neighbourhood

$N_K(e_0)$ of an end e_0 as the set $e_0(K) \cup \{$ ends $e \mid e(K) = e_0(K)\}$.
With this topology $X \cup E(X)$ is a Hausdorff space containing $E(X)$, with its inverse limit topology, as a closed (nowhere dense) subspace.

If $f: X \to Y$ is a continuous proper function (i.e. $F \subset Y$ compact implies $f^{-1}(F)$ compact), then f is extended uniquely and continuously to a function

$$f \cup f_E : X \cup E(X) \to Y \cup E(Y)$$

such that for $e \in E(X)$ and $F \subset Y$ compact $f_E(e) F$ is the (unique) component of $Y \setminus F$ containing $f(e(f^{-1}(F)))$.

Summarizing, we have defined a covariant functor from the category of locally compact, Hausdorff spaces and proper maps into the category of Hausdorff spaces and continuous maps, which assigns to each space X the space $X \cup E(X)$.

It is not difficult to verify that this functor satisfies the following properties:

(1) If $\{X_i \mid i \in A\}$ is the family of connected components of a (locally compact, Hausdorff) space X, then $\{X_i \cup E(X_i) \mid i \in A\}$ is the family of connected components of $X \cup E(X)$;

(2) Let X be locally connected. The function

$$H(X)_K \to H(E(X))_K$$

$$h \mapsto h_E ,$$

is a well defined continuous group homomorphism

Let X be a locally compact, Hausdorff space, and let $K \subset X$ be compact. A connected component V of $X\setminus K$ is said to be <u>bounded</u> if its closure is compact, and otherwise we say that V is <u>unbounded</u>. Define

$$(3)\quad \hat{K} = X\setminus \bigcup \{ V \in C(X\setminus K) \mid V \text{ is unbounded } \} .$$

The proof of the following lemma may be found in Berlanga and Epstein [3, Lemma 9].

1.11 <u>Lemma</u>

Let X be a connected, non-compact, locally connected, locally compact, Hausdorff space. Let $K \subset X$ be a compact subset. Then $X\setminus K$ has only finitely many unbounded components and $\hat{K}$ is compact (see 1.10(3)).

$\square$

1.12 <u>Remark.</u>

It follows that $\{ \hat{K} \mid K \subset X \text{ compact } \}$ is cofinal and hence $E(X)$ is compact. It can be shown that $X \cup E(X)$ is compact and that $E(X)$ is totally disconnected.

It is well known that if X is metric, then $X \cup E(X)$ is metrizable. Also, in this case, the structure of the set of ends $E(X)$ is well known.

1.13 <u>Definitions.</u>

Let X be a locally compact, locally connected, second countable, Hausdorff space, and let $f, g \in H(X)$. We say that f <u>is isotopic to</u> g if there exists a continuous function $h: [o,1] \to H(X)_K$ such that $h_0 = f$ and $h_1 = g$.

Let $f, g \in H_c(X)$. We say that f is <u>compactly isotopic</u> to g if there exists a continuous function $h: [o, 1] \to H_c(X)_{\underrightarrow{\lim}}$ such that $h_o = f$ and $h_1 = g$. It follows that there exists a compact subset K of X such that $h_t \in H(K, X)$ for each $t \in I$, (see Appendix One, statement (5)).

Denote by $H_o(X)$ the set of all homeomorphisms of X isotopic to the identity, and by $H_{c,o}(X)$ the set of all homeomorphisms of X with compact support compactly isotopic to the identity.

Let M be a second countable manifold. Define $H_o^\partial(M)$ to be the path connected component of the identity in $H^\partial(M)_K$, and define $H_{c,o}^\partial(M)$ to be the path connected component of the identity in $H_c^\partial(M)_{\underrightarrow{\lim}}$.

1.14 In 1.4 and 1.7 we studied the continuity of the action of homeomorphisms on measures. Now we turn ourselves to the question of the the transitivity of such action in the case when the undelying space is a manifold.

The von Neumann - Oxtoby - Ulam theorem states that the group of homeomorphisms of a compact connected manifold M acts transitively on the set of " ∂-good" probability measures on M. In [3], Berlanga and Epstein generalized this result to the σ-compact case.

When M is σ-compact the action considered is also transitive on the set of probability measures, but not on the set of infinite measures in general. An example of this phenomenon is the fact that there is no homeomorphism of $< -\infty, \infty >$ onto $< o, \infty >$ preserving Lebesgue measure.

In order to formulate a precise statement, the behaviour of a measure at infinity must first be defined as the above example suggests.

We are also interested in considering the case of non-connected manifolds. Therefore it is not surprising that the following definition involves "ends", "components" and "total mass".

Let M be a second countable manifold, let $\mu \in \mathcal{M}(M)$, and let $\{ M_j \mid j \in A \}$ be the family of connected components of M. Define a function $\alpha(\mu)$ from the set $E(M)$ of ends into the set $[o, \infty]$ of non-negative real numbers union "infinity" such that

$$\alpha(\mu)\, e = \begin{cases} \infty & \text{if } \mu(e(K)) = \infty \text{ for all compact } K \\ \mu(M_j) & \text{if } \mu(M_j) < \infty \text{ and } e(\emptyset) = M_j \\ 0 & \text{if } \mu(M_j) = \infty \text{ , } e(\emptyset) = M_j \\ & \text{and } \mu(e(K)) < \infty \text{ for some compact } K. \end{cases}$$

Example.

Put Lebesgue measure on the intervals $< o, 1 >$, $< o, \infty >$ and $< -\infty, \infty >$. Now consider its disjoint union.

It is easy to verify that if $h \in H(M)$ and $\mu \in \mathcal{M}(M)$ then $\alpha(\mu) \circ h^{-1} = \alpha(h_* \mu)$.

Hence we have a commutative diagram.

$$\begin{array}{ccc} H(M) \times \mathcal{M}(M) & \longrightarrow & \mathcal{M}(M) \\ \downarrow & & \downarrow \\ H(E(M)) \times F(E(M), [o, \infty]) & \rightarrow & F(E(M), [o, \infty]) \end{array}$$

where $F(E(M), [o, \infty])$ denotes the set of functions from $E(M)$

into $[o, \infty]$ (see 1.10(2)).

Let $\mu_o \in M_g^\partial(M)$, and let $M_g^\partial(M,\alpha(\mu_o))$ be the set of all ∂-good Radon measures μ on M such that $\alpha(\mu) = \alpha(\mu_o)$ and $\mu(M_j) = \mu_o(M_j)$

component M_j of M. Note that if a component M_j

ct, then $\alpha(\mu) = \alpha(\mu_o)$ implies $\mu(M_j) = \mu_o(M_j)$.

is isotopic to the identity, then h_ε is the

Therefore $H_0(M)$ acts on $M_g^\partial(M, \alpha(\mu_o))$.

second countable manifold, and let $\mu_o \in M_g^\partial(M)$.

$$\ _g^\partial(M, \alpha(\mu_o)) \longrightarrow M_g^\partial(M, \alpha(\mu_o))$$

$$H_{c,o}^\partial(M) \times M_{g,c}^\partial(M, \mu_o) \longrightarrow M_{g,c}^\partial(M, \mu_o)$$

are transitive.

Proof.

It is enough to prove the theorem for M connected. In this case, the transitivity of the first action is just the main theorem in Berlanga and Epstein [3] . To prove the transitivity of the second action, we can use the results and concepts in [3] as well: Let $\mu \in M_{g,c}^\partial(M, \mu_o)$ and choose a (compact) measure conformable set $K \subset M$ (by [3, Proposition 1 and Lemma 7]) such that into $[o, \infty]$ (see 1.10(2)).

Let $\mu_o \in \mathcal{M}_g^\partial(M)$, and let $\mathcal{M}_g^\partial(M,\alpha(\mu_o))$ be the set of all ∂-good Radon measures μ on M such that $\alpha(\mu) = \alpha(\mu_o)$ and $\mu(M_j) = \mu_o(M_j)$ for each connected component M_j of M. Note that if a component M_j of M is non-compact, then $\alpha(\mu) = \alpha(\mu_o)$ implies $\mu(M_j) = \mu_o(M_j)$.

If $h \in \mathcal{H}(M)$ is isotopic to the identity, then h_ε is the identity on $\mathcal{E}(M)$. Therefore $\mathcal{H}_o(M)$ acts on $\mathcal{M}_g^\partial(M, \alpha(\mu_o))$.

1.15 Theorem

Let M be a second countable manifold, and let $\mu_o \in \mathcal{M}_g^\partial(M)$. Then the actions

$$\mathcal{H}_o^\partial(M) \times \mathcal{M}_g^\partial(M, \alpha(\mu_o)) \longrightarrow \mathcal{M}_g^\partial(M, \alpha(\mu_o))$$

and

$$\mathcal{H}_{c,o}^\partial(M) \times \mathcal{M}_{g,c}^\partial(M, \mu_o) \longrightarrow \mathcal{M}_{g,c}^\partial(M, \mu_o)$$

are transitive.

Proof.

It is enough to prove the theorem for M connected. In this case, the transitivity of the first action is just the main theorem in Berlanga and Epstein [3] . To prove the transitivity of the second action, we can use the results and concepts in [3] as well: Let $\mu \in \mathcal{M}_{g,c}^\partial(M, \mu_o)$ and choose a (compact) measure conformable set $K \subset M$ (by [3, Proposition 1 and Lemma 7]) such that

$$\mu_o(\mathrm{Fr}\,K\ ) = \mu\,(\mathrm{Fr}\,K\ ) = o\ ,$$

$$\mu_o(K) = \mu\,(K)\ ,$$

$$\mu_o\big|_{M\setminus K} = \mu\big|_{M\setminus K}\ ,$$

and apply Proposition 1 in [3], and the remark thereafter to get an $h \in H^{\partial}(K, M)$ with $h_* \mu = \mu_o$ and isotopic to the identity in K.

□

1.16 <u>Lemma.</u>

Let M be a second countable manifold, and let $\mu_o \in M_g^{\partial}(M)$. Then $M_g^{\partial}(M, \alpha\,(\mu_o))_\omega$ is contractible by a contraction fixing μ_o. Furthermore, the contraction can be assumed to induce a continuous map

$$H_c :\ [\,M^{\partial}_{g,c}(M, \mu_o)_{\underset{\rightarrow}{\lim}}\,] \times I \rightarrow M^{\partial}_{g,c}(M, \mu_o)_{\underset{\rightarrow}{\lim}}$$

<u>Proof.</u>

It is not difficult to show that the contraction

$$H : M_g^{\partial}(M,\alpha(\mu_o))_\omega \times I \rightarrow M_g^{\partial}(M, \alpha(\mu_o))_\omega$$

such that

$$H(\mu\,,\,t) = (1 - t)\,\mu + t\,\mu_o$$

is well defined and continuous.

Also, if $K \subset M$ is compact, $\mu \in M_g(K, M, \mu_o)$ and $t \in I$, then $(1 - t)\,\mu + t\,\mu_o \in M^{\partial}_{g}(K, M, \mu_o)$ so H restricts to

$$H_c : M^{\partial}_{g,c}(M, \mu_o) \times I \to M^{\partial}_{g,c}(M, \mu_o) .$$

In principle, the topology we should impose on $M^{\partial}_{g,c}(M, \mu_o) \times I$ to make H_c continuous (following Remark 1.8) is the one coinduced by the coherent family

$$\{ M^{\partial}_{g}(K, M, \mu_o)_{\omega} \times I \mid K \subset M \text{ compact} \} ,$$

but since I is compact, the resulting space, say $[M^{\partial}_{g,c}(M, \mu_o) \times I]_{\underrightarrow{\lim}}$, is homeomorphic to $[M^{\partial}_{g,c}(M, \mu_o)_{\underrightarrow{\lim}}] \times I$.

$\square$

§2. A mapping theorem for topological σ-compact manifolds.

It is the purpose of this section to prove theorem 2.2 . It is a generalization to σ-compact manifolds of a well known result due to M. Brown (see [4]).

Let M be a σ-compact, connected manifold of dimension n. Since the set $\mathcal{E}(M)$ of M is a totally disconnected, compact, metrizable space, we can construct a set E contained in the boundary of the unit cube I^n in such a way that E is homeomorphic to $\mathcal{E}(M)$. Now $I^n \setminus E$ and M are two manifolds with the same set of ends. Theorem 2.2 says that M is the identification space obtained from $I^n \setminus E$ by identifying certain points of $\partial I^n \setminus E$ to certain other points of $\partial I^n \setminus E$.

2.1 Definitions

Let X be a subset of a topological space Y. We define $\overset{\circ}{X}$ to be the topological interior of X in Y. Recall that $\mathrm{Fr}\, X$ is defined as the topological frontier of X in Y and $\mathrm{Cl}\, X$ as the closure of X in Y (see 1.1 and 1.5). Call X a (closed) n-cell if X is homeomorphic to the unit n-cube $I^n = [o, 1]^n$. For a subset X of a manifold M we define $\mathrm{Int}\, X$ to be $(M \setminus \partial M) \cap \overset{\circ}{X}$.

A subset X of an n-manifold M is cellular if for every neighbourhood U of X there is an n-cell Q such that $X \subset \mathrm{Int}\, Q \subset Q \subset U$. An (n-1)-manifold B^{n-1} is bicollared in M if there is a homeomorphism P of $B \times \langle -1, 1 \rangle$ onto a neighbourhood of B in M such that $P(b, o) = b$, for all $b \in B$. If B is closed in M we require also that P can be extended to a closed embedding of $B \times [-1, 1]$ onto M. If $\delta_i : B \to \langle o, 1 \rangle$ $(i=1,2)$ continuous are given, we define

$$B \times \langle\langle -\delta_1, \delta_2 \rangle\rangle = \{ (b,t) \mid -\delta_1(b) < t < \delta_2(b) \} .$$

If $i = 1$ or $i = 2$ is given, we define

$$B \times \{\{ (-1)^i \delta_i \}\} = \{ (b,t) \mid (-1)^i \delta_i(b) = t \} .$$

Finally, we define

$$B \times [[-\delta_1, \delta_2]] = B \times \langle\langle -\delta_1, \delta_2 \rangle\rangle \cup \bigcup_{i=1}^{2} B \times \{\{(-1)^i \delta_i\}\}.$$

If B is the boundary of an n-dimensional submanifold C^n of M^n, then $B \times \langle -1, o]$ and $B \times [o, 1 \rangle$ denote the inner and outer collars of B. In general, we will not distinguish $(b,t) \in B \times \langle -1,1 \rangle$ from $P((b,t))$.

2.2 Theorem.

Let M^n be a connected, second countable manifold. Then there exists a compact set $E \subset \partial I^n$ and a continuous proper surjection $\varphi : I^n \setminus E \to M$ such that

(1) $\varphi|_{\mathrm{Int}\, I^n} : \mathrm{Int}\, I^n \to \varphi(\mathrm{Int}\, I^n)$

is a homeomorphism.

(2) $\varphi(\mathrm{Int}\, I^n) \cap \varphi(\partial I^n \setminus E) = \emptyset$;

(3) φ extends naturally to $\tilde{\varphi} : I^n \to M \cup \mathcal{E}(M)$ in such a way that $\tilde{\varphi}|_E$ is a homeomorphism from E onto $\mathcal{E}(M)$.

Furthermore, if $n \geq 2$ then E can be chosen to be contained in $[\,1/3, 2/3\,] \times \{(1/2, 1/2, \ldots, 1/2, 1)\}$. In particular, if E has no isolated points, then E is a "standard" copy of the Cantor set in ∂I^n .

In order to prove this theorem, we first prove three lemmas.

2.3 Lemma. (cf. M. Brown [4])

Let M^n be a manifold with $n \geq 3$ and let d be a metric on M. Let C^n be a closed n-dimensional manifold with bicollared boundary ∂C in M.

Let $\varepsilon > 0$ and a continuous function $\delta : \partial C \to \langle 0, 1\rangle$ be given. Suppose $\Lambda = \{E_j\}_{j \in J}$ is an (at most countable) locally finite family of sets in M such that each E_j is a closed n-cell of diameter less than $\varepsilon/2$ whose interior intersects C. Let $X = \{x_i\}_{i \in L}$ be a locally finite set of points in $\bigcup_{j\in J} \operatorname{Int} E_j \setminus C$. Then there is a locally finite set of points $X' = \{x_i'\}_{i\in L}$ in $\partial C \times \langle\langle o, \delta\rangle\rangle$ and an $\varepsilon/2$-homeomorphism $h: M \to M$ such that $\operatorname{supp} h \subset \bigcup_{j\in J} \operatorname{Int} E_j \setminus C$ and $h(x_i') = x_i$ for each $i \in L$.

Proof.

We may assume, without loss of generality, that $x_{i_1} \neq x_{i_2}$ for $i_1 \neq i_2$ (hence L is at most countable).

Associate with each x_i some element, say $E_{j(i)}$, of Λ which contains x_i in its interior. Associate with each E_j a point $y_j \in C \cap \operatorname{Int} E_j$. For $i \in L$ let α_i be a polygonal arc (relative to some combinatorial structure on $E_{j(i)}$) in $\operatorname{Int} E_{j(i)}$ from x_i to $y_{j(i)}$. Since an n-dimensional connected manifold cannot be disconnected by a subset of dimension less than or equal to $n-2$ (see Hurewicz and Wallman [14, Theorem IV. 4, p.48]), this can be done in such a manner that α_{i_1} and α_{i_2} are disjoint or intersect only in the common end point

$$y_{j(i_1)} = y_{j(i_2)} .$$

Let x_i' be a point of $\alpha_i \cap \partial C \times \langle\langle o, \delta\rangle\rangle$ such that the segment $[x_i, x_i']$ of α_i does not intersect C. Since α_i is polygonal in $E_{j(i)}$, so is $[x_i, x_i']$. Hence $[x_i, x_i']$ is cellular in $E_{j(i)}$ and hence cellular in M. Hence there exists a (locally finite) family $\{Q_i\}_{i\in L}$ of n-cells such that

(1) $Q_i \cap Q_j = \emptyset$ if $i \neq j$;

(2) $[x_i, x'_i] \subset \mathring{Q}_i$;

(3) $Q_i \cap C = \emptyset$;

(4) $Q_i \subset \mathrm{Int}\, E_{j(i)}$.

Let h be a homeomorphism of M onto M such that

(1) $h | (M \setminus \bigcup_{i \in L} Q_i) = \mathrm{Id}$;

(2) $h(Q_i) = Q_i$;

(3) $h(x'_i) = x_i$.

Then h is the required homeomorphism.

□

2.4 Lemma.

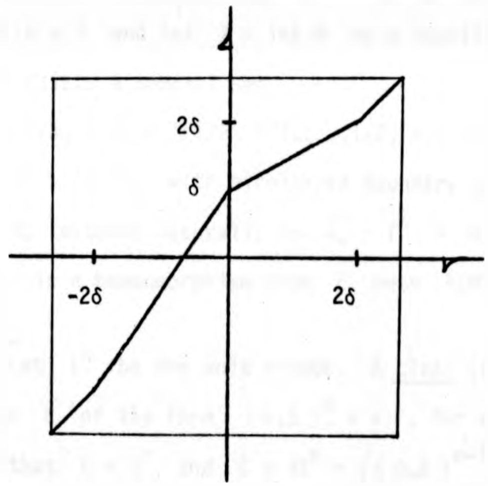
Suppose that $0 < \gamma < 1$ and that the hypotheses of the above lemma are satisfied. Then there is an ε-homeomorphism f of M onto M such that $f(C) \supset f(\mathring{C}) \supset C \cup X$ and $\mathrm{supp}\, f \subset (\bigcup \mathrm{Int}\, E_i \setminus C) \cup \partial C \times \langle -\gamma, \gamma \rangle$. In particular, f fixes pointwise the "inner" n-manifold bounded by $\partial C \times \{-\gamma\}$.

Proof.

Choose $\delta : \partial C \to \langle 0, \gamma/2 \rangle$ continuous and such that for each $c \in \partial C$ the diameter (with respect to the induced metric) of $\{c\} \times [-2\delta(c), 2\delta(c)]$ - in the collar $C \times [-1, 1]$ - is less than $\varepsilon/2$.

Let $\alpha : \partial C \to H([-1, 1])$ be defined by the formula

$$\alpha_c(t) = \begin{cases} t & -1 \le t \le -2\,\delta(c) \\ (3/2)t + \delta(c) & -2\delta(c) \le t \le 0 \\ (1/2)t + \delta(c) & 0 \le t \le 2\delta(c) \\ t & 2\delta(c) \le t \le 1 \end{cases} ;$$



Since $\partial C \times [-1, 1]$ is closed in M, we can define a homeomorphism $g \in H(M)$ such that g is the identity outside $\partial C \times \langle -1, 1 \rangle$ and is given by

$$g(c,t) = (c, \alpha_c(t))$$

for each $(c,t) \in \partial C \times [-1, 1]$.

Therefore, g is fixed on the manifold bounded by $\partial C \times \{\{-2\delta\}\}$, stretches $\partial C \times \{0\}$ parametrically onto $\partial C \times \{\{\delta\}\}$ and is fixed outside $\partial C \times \{\{2\delta\}\}$. Furthermore, g is an $\varepsilon/2$-homeomorphism and if h is the homeomorphism obtained in the conclusion of the above lemma, then $f = h \circ g$ is the required ε-homeomorphism.

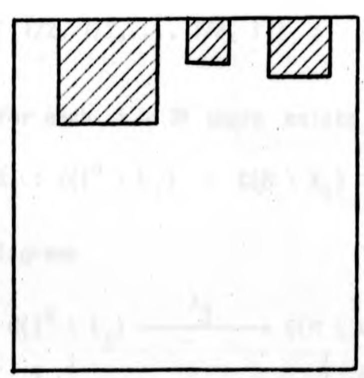
□

2.5 Lemma

Let M^n be a connected, second countable, n-dimensional manifold with $n \geq 3$ and let $X \subset \text{Int } M$ be a locally finite set of points. Then there exists a compact set
$E \subset [1/3, 2/3] \times \{(1/2, 1/2, \ldots, 1/2, 1)\} \subset \partial I^n$, and a proper embedding $\varphi_* : I^n \setminus E \hookrightarrow M$ with bicollared boundary such that $\varphi_*(\text{Int } I^n) \supset X$ and φ_* extends naturally to $\overline{\varphi_*} : I^n \to M \cup E(M)$ in such a way that $\overline{\varphi_*}|_E$ is a homeomorphism from E onto $E(M)$.

Proof.

Let I^n be the unit n-cube. A clean (closed) n-cube in I^n is a cube C of the form $[0,\beta]^n + \vec{v}$, for some $\beta > 0$ and $\vec{v} \in \mathbb{R}^n$, such that $C \subset I^n$ and $C \cap \partial I^n = ([0,\beta]^{n-1} \times \{\beta\}) + \vec{v}$.



Observe that if $C_1,\dots, C_k$ is a disjoint collection of clean cubes then $Cl(I^n \setminus \cup C_i)$ is homeomorphic to I^n. We divide the proof in three steps.

Step 1

Let $\emptyset \neq \mathring{K}_0 \subset K_0 \subset \mathring{K}_1 \subset K_1 \subset,\dots, \subset M$ be a collection of compact sets such that

(1) $M = \bigcup_{i=0}^{\infty} K_i$,

(2) $K_i = \hat{K}_i$.

It is not difficult now to define a sequence $\emptyset \neq \mathring{L}_0 \subset L_0 \subset \mathring{L}_1 \subset \dots \subset I^n$ of n-cells such that

(a) The complement of $\mathring{L}_i$ is the finite disjoint union of clean cubes of diameter less or equal 2^{-i}, and such that, for each $A \in C(I^n \setminus L_i)$, $A \cap [1/3, 2/3] \times \{(1/2, 1/2,\dots,1/2, 1)\} \neq \emptyset$.

Hence, $E = \bigcap_i I^n \setminus L_i$ is contained in $[1/3, 2/3] \times \{(1/2, 1/2,\dots, 1/2, 1)\}$;

(b) For each $i \in \mathbb{N}$ there exists a bijection

$$\lambda_i : C(I^n \setminus L_i) \to C(M \setminus K_i)$$

such that the diagrams

$$\begin{tikzcd} C(I^n \setminus L_j) \arrow[r, "\lambda_j"] \arrow[d, "\rho_i^j"'] & C(M \setminus K_j) \arrow[d, "\rho_i^j"] \\ C(I^n \setminus L_i) \arrow[r, "\lambda_i"] & C(M \setminus K_i) \end{tikzcd}$$

commute $(i < j)$.

The reader can readily verify the following assertion:

Assertion:

$$E = \bigcap_{i \in \mathbb{N}} I^n \setminus L_i \quad , \; E(I^n \setminus E) \text{ and } E(M) \text{ are homeomorphic.}$$

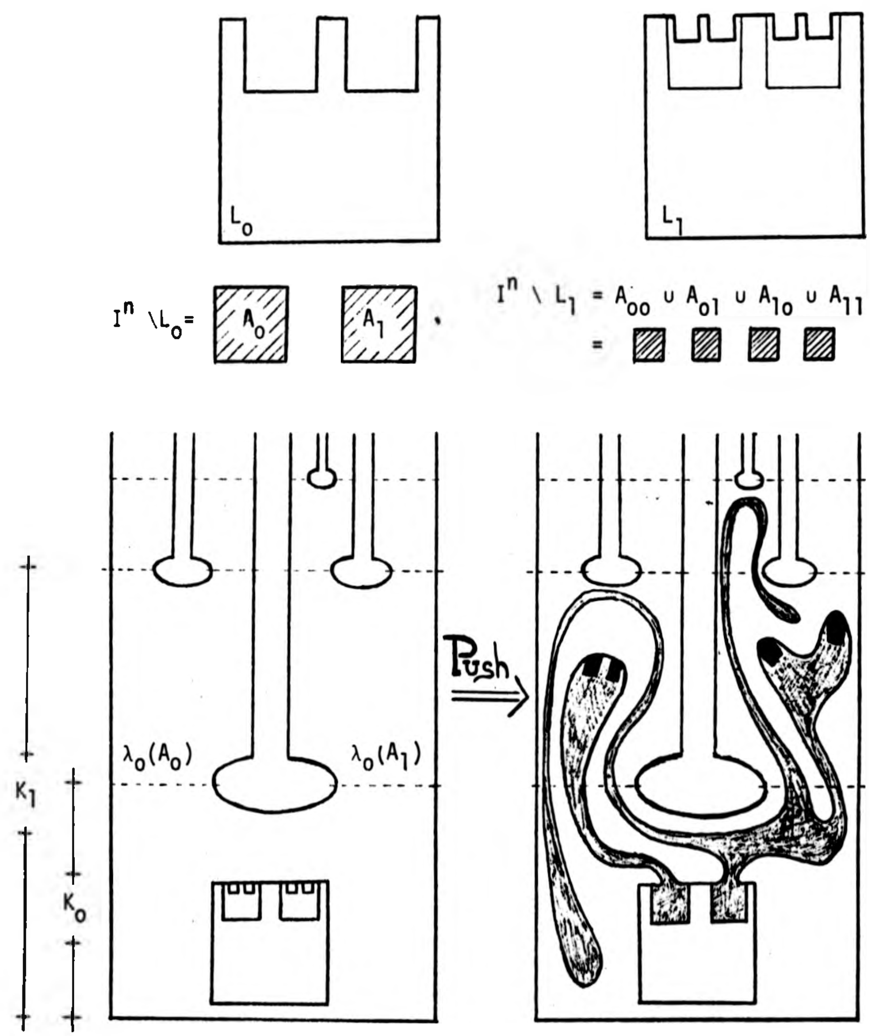
Furthermore, the identity map $I^n \setminus E \hookrightarrow I^n \setminus E$ extends naturally to a homeomorphism of $I^n = (I^n \setminus E) \cup E$ onto $(I^n \setminus E) \cup E(I^n \setminus E)$.

□

Remarks

We would like now to embed a copy of I^n in $\mathring{K}_0$ and start an inductive process with the aid of the combinatorial scheme constructed above. Suppose for a moment that I^n is actually contained in $\mathring{K}_0$ and that A_0 is a component of $I^n \setminus L_0$. Then, we want to "push" A_0 (or some part of A_0) to where it corresponds. That is, into $\lambda_0(A_0)$. Now let A_{00} be a component of $I^n \setminus L_1$ contained in A_0. A further "push" should take A_{00} (or a part of A_{00}) into $\lambda_1(A_{00})$. And so on.

Many things can go wrong in the process. The following diagram intends to show some of the difficulties.



There is nothing wrong with the "push" we gave to A_0, but A_1 is so badly deformed that we cannot push, say A_{00}, any further (and achieve convergence at the end of the story).

The situation for A_{1o} is bad as well. Certainly we can push it once more as to move some part of it into $\lambda_1(A_{1o})$. But then a third push most probably will be impossible.

Although the example is two-dimensional (we are assuming dimension no less than three in this lemma), it is easy to extend it to dimension three (the "tentacles" will now look like "domes").

There can be problems with the compact sets $K_0 \subset K_1 \subset K_2 \subset \ldots \subset M$ as well, so we will need to assume some "connectivity" properties.

It is not difficult to start Step 1 instead with a sequence

$$\emptyset \neq \mathring{K}_0 \subset \hat{K}_0 \subset \mathring{K}_1 \subset \hat{K}_1 \subset \mathring{K}_2 \subset \hat{K}_2 \subset \ldots \subset M$$

such that

(1) $\mathring{K}_i$ is connected

(2) $M \setminus \hat{K}_i$ has exactly the same number of components as $\mathring{K}_{i+1} \setminus \hat{K}_i$.

We now proceed into Step 2 of this lemma.

<u>Step 2.</u>

Let $\varphi_0 : I^n \hookrightarrow \mathring{K}_0$ be an embedding with bicollared boundary. Then there exists $h \in \mathcal{H}_c(M)$ such that (see 1.10 and Step 1(b)),

(1) $\operatorname{supp} h_1 \cap \varphi_0(L_0) = \emptyset$;

(2) $\operatorname{supp} h_1 \subset \mathring{K}_1$;

(3) If $A \in \mathcal{C}(I^n \setminus L_1)$ then

a) $h_1 \circ \varphi_0(A) \subset \lambda_0 \circ \rho_0^1(A)$;

b) $h_1 \circ \varphi_0(A)$ and $\lambda_1(A)$

are not separated in M by $h_1 \circ \varphi_0(I^n \setminus A) \cup \hat{K}_0$ (that is, $h_1 \circ \varphi_0(A)$ and $\lambda_1(A)$ lie in the same connected component of $M \setminus (h_1 \circ \varphi_0(I^n \setminus A) \cup \hat{K}_0)$.

Proof.

Let $A_1, A_2, \ldots, A_k$ be the components of $I^n \setminus L_1$. It is not difficult to construct a family of disjoint arcs, say $\{ \gamma_i : [0,2] \to M \mid 1 \le i \le k \}$ and a family $\{ U_i \mid 1 \le i \le k \}$ of disjoint connected open sets in $\mathring{K}_1$ such that, for each i,

(1) $U_i \cap \varphi_0(I^n) \subset \varphi_0(A_i)$

(2) $\gamma_i([0,1]) \subset U_i$

(3) $\gamma_i([1,2]) \subset \mathring{K}_2 \setminus \hat{K}_0$

(4) $\gamma_i(0) \in \varphi_0(A_i)$

(5) $\gamma_i(1) \in \lambda_0 \circ \rho_0^1(A_i)$

(6) $\gamma_i(2) \in \lambda_1(A_i)$

(this can be done because $M \setminus \varphi_0(I^n)$ is connected and an n-dimensional manifold cannot be disconnected by a set of dimension $n-2$. See Hurewicz and Wallman [14 , Theorem IV. 4, p.48]).

Since $\mathcal{H}_c(U_i) \subset \mathcal{H}_c(M)$ acts transitively on U_i, we can find a homeomorphism $h_{1,i} \in \mathcal{H}_c(U_i)$ such that $h_{1,i}(\gamma_i(0)) = \gamma_i(1)$.

For each $i = 1,2,\ldots, k$, let $\tau_i \in [1,2]$ be the last parameter such that its image under γ_i lies in $h_{1,i} \circ \varphi_0(I^n)$.

Consequently, there is a unique x_i in $\partial I^n \cap A_i$ with $\gamma_i(\tau_i) = h_{1,i} \circ \varphi_0(x_i)$. Now choose a clean closed cube B_i such that $x_i \in B_i \subset A_i$ and $h_{1,i} \circ \varphi_0(B_i) \subset \lambda_0 \circ \rho_0^1(A_i)$.

With a homeomorphism of M sending $\varphi_0(I^n)$ onto itself and supported in a small neighbourhood of $\varphi_0(\mathrm{Cl}\ A_i)$ we can shrink $\varphi_0(\mathrm{Cl}\ A_i)$ onto $\varphi_0(B_i)$ before applying $h_{1,i}$. Therefore, without loss of generality we can assume that $A_i = B_i$ and that $\mathrm{supp}\ h_{1,i} \cap \mathrm{supp}\ h_{1,j} = \emptyset$ for $i \neq j$. Hence,

$$h_{1,i} \circ \varphi_0(A_i) \subset \lambda_0 \circ \rho_0^1(A_i)$$

$$h_{1,i} \circ \varphi_0(A_i) \quad \text{and} \quad \lambda_1(A_i)$$

are not separated in M by $h_{1,i} \circ \varphi_0(I^n \setminus A_i)$.

Finally, the homeomorphism

$$h_1 = h_{1,1} \circ h_{1,2} \circ \cdots \circ h_{1,k}$$

has the required properties.

$\square$

<u>Step 3</u>

By induction, we can construct a sequence $\{h_i\}_{i \in \mathbb{N}}$ of homeomorphisms with compact support such that, for each $i = 0,1,\ldots$

(1) supp $h_{i+1} \cap (h_i \circ h_{i-1} \circ \dots \circ h_1 \circ \varphi_0(L_i)) = \emptyset$;

(2) supp $h_{i+1} \subset \mathring{K}_{i+1}$

(3) supp $h_{i+1} \cap \hat{K}_{i-1} = \emptyset$;

(4) If $A \in C(I^n \setminus L_{i+1})$, then

(a) $h_{i+1} \circ h_i \circ \dots \circ h_1 \circ \varphi_0(A) \subset \lambda_i \circ \rho_i^{i+1}(A)$;

(b) $h_{i+1} \circ h_i \circ \dots \circ h_1 \circ \varphi_0(A)$ and $\lambda_{i+1}(A)$

are not separated in M by $h_{i+1} \circ h_i \circ \dots \circ h_1 \circ \varphi_0(I^n \setminus A) \cup \hat{K}_i$.

Define $\varphi_i = h_i \circ \dots \circ h_1 \circ \varphi_0$, $i = 1,2,\dots$.

Therefore, the following properties hold:

(5) $\varphi_i | L_i = \varphi_{i+k} | L_i$ all $i, k \in \mathbb{N}$;

(6) $\varphi_{i+k}(A) \subset \lambda_i \circ \rho_i^{i+1}(A)$ all $i \in \mathbb{N}$, $k \in \mathbb{N}\setminus\{o\}$,

and all $A \in C(I^n \setminus L_{i+1})$. It follows that $\lim_{i\to\infty} \varphi_i = \varphi_*$ exists in $\bigcup_{i=o}^{\infty} L_i$ and is such that

(7) $\varphi_* | L_i = \varphi_i | L_i$ all $i \in \mathbb{N}$;

(8) $\varphi_*(A) \subset \lambda_i \circ \rho_i^{i+1}(A)$ all $i \in \mathbb{N}$ and all $A \in C((I^n \setminus E) \setminus L_{i+1})$;

(9) $\varphi_*^{-1}(\hat{K}_i) \subset L_{i+1}$.

Property (7) says that φ_* is continuous and injective. Property (9) (which follows from (8)) tells us that $\varphi_* : I^n \setminus E \to M$ is proper, and therefore induces a map

$$\varphi_* \cup \varphi_\varepsilon : (I^n \setminus E) \cup E(I^n \setminus E) = I^n \to M \cup E(M)$$

such that if e is an end of $I^n \setminus E$, $\varphi_\varepsilon(e)\,\hat{K}_i$ is the component of $M \setminus \hat{K}_i$ containing $\varphi_*(e(\varphi_*^{-1}(\hat{K}_i)))$, hence (by (9)) it is equal to the component of $M \setminus \hat{K}_i$ containing $\varphi_*(e(L_{i+1}))$, but (by (8)) this is just $\lambda_i \circ \rho_i^{i+1}(e(L_{i+1})) = \lambda_i(e(L_i))$.

That is, we have proved that the following diagram

$$\begin{tikzcd}
E(I^n \setminus E) \arrow[r, "\Pi_i"] \arrow[d, dash, "\varphi_\varepsilon"'] & C(I^n \setminus L_i) \arrow[d, "\lambda_i"] \\
E(M) \arrow[r, "\Pi_i"] & C(M \setminus \hat{K}_i)
\end{tikzcd}$$

commutes. Since each λ_i is bijective, φ_ε must be a homeomorphism.

Therefore, we have constructed a proper embedding

$$\varphi_* : I^n \setminus E \hookrightarrow M$$

inducing a homeomorphism on ends.

In order to complete the proof of Lemma 2.5 we need to produce a bicollar of $\varphi_*(\partial I^n \setminus E)$ and we need to "expand" the image C of $I^n \setminus E$ in M as to contain X in its interior.

Let E' be the projection of E into I^{n-1}, so $E = E' \times \{1\}$.

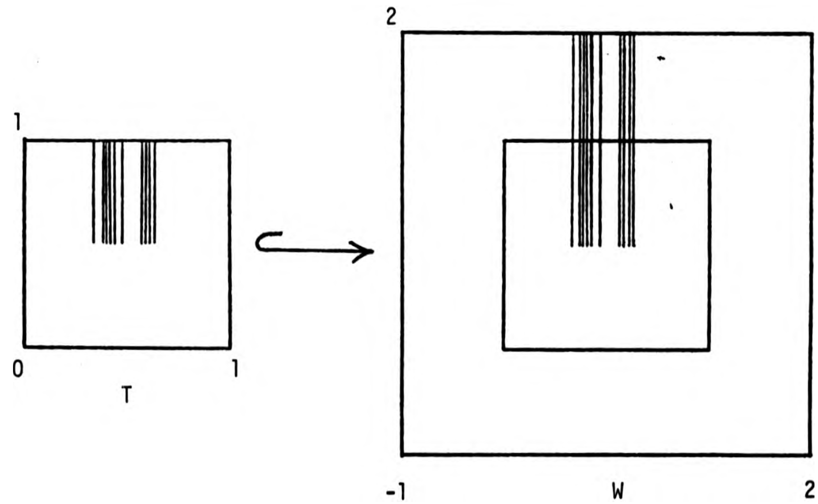
It is not difficult to show that the spaces

$$W = [-1,2]^n \setminus E' \times [1/2, 2]$$

and

$$T = W \cap I^n$$

are homeomorphic to $I^n \setminus E$ and that the inclusion map $T \hookrightarrow W$ is a proper map inducing a homeomorphism on ends.



Therefore, without loss of generality, we can assume that the domain of the map φ_* is W. But now $\varphi_*|_T$ has the same properties of φ_* with the advantage that ∂T has a natural bicollar contained in W.

It now only remains to "expand" the image of φ_*. To this purpose we can construct a locally finite family $\Lambda_0 = \{E_j\}_{j \in J}$ of closed n-cells such that

$$X \setminus C \subset \bigcup_{j \in J} \operatorname{Int} E_j$$

$$\operatorname{Int} E_j \cap C \neq \emptyset \qquad \text{all} \quad j \in J .$$

Therefore, by an application of Lemma 2.4, say with $\gamma = 1/2$ and $\varepsilon = \infty$, we get the desired expansion.

□

<u>Proof of Theorem</u> 2.2 <u>when</u> $\dim M^n \geq 3$.

Let d be a complete metric on M.

Let $\Lambda_1, \Lambda_2, \ldots$ be a sequence of locally finite covers of M such that each element of Λ_i is a closed n-cell of diameter less than 2^{-i-1} and $\operatorname{Int} M = \bigcup_{E \in \Lambda_i} \operatorname{Int} E$.

For each i, let X_i be a locally finite set of points such that

(1) $\operatorname{Int} E \cap X_i \neq \emptyset$ if $E \in \Lambda_i$;

(2) $X_i \subset \operatorname{Int} M$.

Let C_1 be the image under φ_* where φ_* is the embedding given by the above lemma, and assume that $X_1 \subset \operatorname{Int} C_1$. Applying Lemma 2.4 with $X = X_2 \setminus C_1$, $\Lambda = \Lambda_1$ and γ small, we get a $\frac{1}{2}$-homeomorphism f_1 of M onto itself such that

$$M \supset C_2 = f_1(C_1) \supset f_1(\mathring{C}_1) \supset C_1 \cup X_2 \ ;$$

$$f_1 \big|_{(1-\gamma)\, C_1} = \mathrm{Id} \ ;$$

where $(1-\gamma)\, C_1 = C_1 \setminus \partial C_1 \times \langle -\gamma, 0\,]$.

Repeated applications of 2.4 give a sequence $f_1, f_2, \ldots$ of homeomorphisms of M onto itself such that for each $m \in \mathbb{N} \setminus \{0\}$,

(1) f_m is a $(\frac{1}{2})^m$ - homeomorphism ;

(2) $$M \supset f_m \circ \ldots \circ f_1(C_1) \supset f_m \circ \ldots \circ f_1(\mathring{C}_1) \supset C_1 \cup \bigcup_{i=1}^{m+1} X_i \quad ;$$

(3) $$f_{m+1} \mid f_m \circ \ldots \circ f_1((1-\gamma/2^m)\, C_1) = \mathrm{Id}\ .$$

Clearly $f_m \circ \ldots \circ f_1$ converges to a map φ such that

$$\varphi(C_1) = \lim_{m \to \infty} f_m \circ \ldots \circ f_1(C_1) = M\ ;$$

φ is a homeomorphism on $\mathring{C}_1$;

$$\varphi^{-1} \circ \varphi(\partial C_1) = M \setminus \mathring{C}_1 \quad ;$$

so that when φ is restricted to C_1 we get the required map.

□

2.6 <u>Remark</u>

When the dimension of the manifold M is less or equal two, Theorem 2.2 follows from the clasification theorem for second countable manifolds of dimensions one and two, (see Ahlfors and Sario [1]).

§3. Does the action of homeomorphisms on ∂-good measures have a continuous section?

Theorem 1.15 states that, for a fixed measure $\mu_o \in M_g^\partial(M)$ on a second countable manifold M, the continuous map

$$\pi_o : H_o^\partial(M)_\kappa \longrightarrow M_g^\partial(M, \alpha(\mu_o))_\omega$$

such that

$$h \mapsto h_* \mu_o$$

is surjective.

In [9] , A. Fathi poses the following question in the case where M is compact.

Question.- Does the (surjective) map $\pi_o : H_o^\partial(M)_\kappa \longrightarrow M_g^\partial(M, \alpha(\mu_o))_\omega$ have a continuous section?

The answer to this question may be negative.

A. Fathi, inspired in the methods of Oxtoby and Ulam (see [20]), gave a partial positive answer to the above question.

In the paragraphs that follow, we generalize the work of A. Fathi to the σ-compact case.

Firstly, we restrict our considerations to μ_o-biregular homeomorphisms of M and to μ_o-biregular measures on M. Such homeomorphisms and measures are certainly required to preserve the "behaviour of μ_o at infinity" as well.

We encounter the difficulty that the weak topology on $M_g^\partial(M, \alpha(\mu_o))$ does not have "control over the μ_o-finite ends of M". Accordingly, it is necessary to strengthen the topology of the space of measures.

Finally, in this restricted context of "e-biregular" homeomorphisms and measures, we are able to prove that the map $\pi_o : h \mapsto h_*\mu_o$ has a continuous section.

This result will enable us to imply, in §4, that some measure preserving groups of homeomorphisms are locally contractible.

The following definitions and the theorem thereafter make precise the above discussion.

3.1 <u>Definitions.</u>

Let M be a second countable manifold, and let $\mu_o \in M_g^\partial(M)$.

Define $H^{\partial}(M, \mu_o\text{-e-reg})$ to be the group of all homeomorphisms h in $H^{\partial}(M)$ such that

(1) h preserves the behaviour of μ_o at infinity (i.e. $\alpha(h_*\mu_o) = \alpha(\mu_o)$, see 1.14);

(2) $h_* \mu_o(M_j) = \mu_o(M_j)$ for each connected component M_j of M ;

(3) $h_*\mu_o$ and μ_o have the same sets of measure zero.

Define $M_g^{\partial}(M, \mu_o\text{-e-reg})$ to be the set of all measures μ in $M_g^{\partial}(M)$ such that

(1)' μ and μ_o agree on ends, (i.e. $\alpha(\mu) = \alpha(\mu_o)$) ;

(2)' $\mu(M_j) = \mu_o(M_j)$ for each connected component M_j of M.

(3)' μ and μ_o have the same sets of measure zero.

Observe that condition (2) and condition (2)' are implied by condition (1) and (1)' respectively, in the case where each connected component M_j of M is non-compact.

Let K be a compact subset contained in M. Define $H^{\partial}(K, M, \mu_o\text{-e-reg})$ to be the group of all homeomorphisms h in $H^{\partial}(M, \mu_o\text{-e-reg})$ such that its support is contained in K.

Define $M_g^{\partial}(K, M, \mu_o\text{-e-reg})$ to be the set of all measures μ in $M_g(M, \mu_o\text{-e-reg})$ such that μ is K-related to μ_o.

Finally, define

$$H^{\partial}_{c}(M,\ \mu_o\text{-e-reg}) = H^{\partial}_{c}(M) \cap H^{\partial}(M,\ \mu_o\text{-e-reg})$$

$$M^{\partial}_{g,c}(M,\ \mu_o\text{-e-reg})$$

$$= M^{\partial}_{g,c}(M,\ \mu_o) \cap M^{\partial}_{g}(M,\ \mu_o\text{-e-reg})$$

Endow $H^{\partial}(M,\ \mu_o\text{-e-reg})$ with the compact open topology (denoted by κ) and $H^{\partial}_{c}(M,\ \mu_o\text{-e-reg})$ with the topology coinduced by the coherent family $\{ H^{\partial}(K,\ M,\ \mu_o\text{-e-reg})_{\kappa} \mid K \subset M$ compact $\}$.

Endow $M^{\partial}_{g}(M,\ \mu_o\text{-e-reg})$ with the weak topology (denoted by ω) induced by the family of continuous functions $f : M \to \mathbb{R}$, with compact support. Endow $M^{\partial}_{g,c}(M,\ \mu_o\text{-e-reg})$ with the topology coinduced by the coherent family

$$\{ M^{\partial}_{g}(K,\ M,\ \mu_o\text{-e-reg}) \mid K \subset M \text{ compact } \}.$$

The following example intends to explain why we want to change the topology on the space of e-biregular measures.

Example.

Let $X = [\ o,\ \infty >$, $Y = <o,\ \infty >$, and let dt denote Lebesgue measure on both spaces.

Although $M^{\partial}_{g}(X,\ dt\text{-e-reg})$ and $M_{g}(Y,\ dt\text{-e-reg})$ are equal as sets, their corresponding weak topologies are different. Consider the following two functions,

$$M_g^{\partial}(X,\ \text{dt-e-reg})_\omega \to \mathbb{R}$$

$$\mu \mapsto \mu([\,o,\ 1\,])$$

and

$$M_g(Y,\ \text{dt-e-reg})_\omega \to \mathbb{R}$$

$$\mu \mapsto \mu(<o,\ 1\,])$$

Lemma 1.5 implies that $\mu \mapsto \mu(\,[o,\ 1]\,)$ is continuous, but now it is easy to see that the second function is not. Indeed, define for each $n = 1,2,\ldots,$ the measure

$$\mu_n = dt + n\cdot dt\big|_{<o,\ 1/n]} \quad .$$

Then, it is readily verified that $\mu_n \rightharpoonup dt$ in $M_g(Y,\ \text{dt-e-bireg})_\omega$ but $\mu_n(<o,\ 1\,]) = 2$ for each $n = 1,2,\ldots$.

Let $\mu \in M_g(Y,\ \text{dt-e-reg})$.

Suppose now that we want to produce a homeomorphism $\sigma(\mu)$ on Y such that $\sigma(\mu)_*\, dt = \mu$. Roughly speaking, a necessary condition is to check the number $\mu(<o,\ 1\,])$ in order to decide how much "mass" we have to "pump" in or out $<o,\ 1\,]$, via $\sigma(\mu)$, to equate $\sigma(\mu)_*\, dt(<o,1\,])$ with $\mu(<o,\ 1\,])$. With the weak topology on $M_g(Y,\ \text{dt-e-reg})$ this process does not depend continuously on μ. This concludes the example.

Now let M be a second countable manifold, and let $\mu_o \in M_g^{\partial}(M)$.

Then μ_o splits the ends of M into two disjoint subsets.

(1) The set $E_F(M)$ of μ_o-finite ends of M defined to be the set of all ends $e \in E(M)$ such that there exists a compact subset $K \subset M$ for which $\mu_o(e(K))$ is finite (i.e. $E_F(M) = \alpha(\mu_o)^{-1}([0, \infty\rangle))$.

(2) The set $E_I(M)$ of μ_o-infinite ends of M defined to be the set $\alpha(\mu)^{-1}(\{\infty\})$.

By the definition of $\alpha(\mu_o)$ the set of μ_o-finite ends is open, hence the space $M \cup E_F(M)$ is a locally compact (locally connected, second countable) space. Incidentally, the ends of $M \cup E_F(M)$ are just the ends in $E_I(M)$.

The inclusion $i : M \hookrightarrow M \cup E_F(M)$ induces an injection

$$i_* : M_g^\partial(M, \alpha(\mu_o)) \hookrightarrow M_g(M \cup E_F(M))$$

We define the $e\,\omega$-topology on $M_g^\partial(M, \mu_o\text{-e-reg})$ to be the topology this set gets as a subspace of $M_g(M \cup E_F(M))_\omega$. In other words, the $e\,\omega$-topology is the weak topology induced by the family of continuous functions $f: M \cup E_F(M) \to \mathbb{R}$ with compact support.

The following facts are easily verified.

(1) $H^\partial(M, \mu_o\text{-e-reg})$ endowed with the compact open topology acts continuously on $M_g^\partial(M, \mu_o\text{-e-reg})_{e\,\omega}$ (c f. Theorem 1.4).

(2) For each $K \subset M$ compact, the ω-topology and the $e\,\omega$-topology agree on $M_g^\partial(K, M, \mu_o\text{-e-reg})$.

We can state now the main theorem of the present section.

3.2 Theorem.

Let M be a second countable manifold, and let $\mu_o \in M_g^\partial(M)$. Let

$$\pi_o : H^\partial(M,\ \mu_o\text{-e-reg})_\kappa \to M_g^\partial(M,\ \mu_o\text{-e-reg})_{e\omega}$$

be defined by

$$\pi_o\,(h) = h_* \mu_o\ .$$

Then π_o has a continuous section σ .

Furthermore, σ restricts continuously to

$$\sigma_c : M_{g,c}^\partial(M,\ \mu_o\text{-e-reg})_{\underset{\to}{\lim}} \to H_c^\partial(M,\ \mu_o\text{-e-reg})_{\underset{\to}{\lim}}$$

The method of proof of this theorem is first to reduce the general case to the case in which $M^n = I^n \setminus E$, where $E \subset \partial I^n$ is a totally disconnected compact set (as in Theorem 2.2), and μ_o is any (good) measure on $I^n \setminus E$ having the same sets of measure zero as standard Lebesgue measure on I^n (restricted to $I^n \setminus E$).

Accordingly, we start with two results that allow us to make such reduction. Then we proceed into Lemma 3.5, which is the main technical result of the present discussion : It starts with the space $I^n \setminus E$, a compact subset L contained in $I^n \setminus E$ and two e-biregular measures, say μ_p and ν_p, depending continuously on a parameter p. Roughly speaking, Lemma 3.5 states that there is a homeomorphism f_p, depending continuously on p, which sends $\mu_p|_L$ into $\nu_p|_L$ and "pumps" the right amount of mass into each component of $(I^n \setminus E) \setminus L$.

An induction over an increasing sequence of compact sets whose union is $I^n \setminus E$ completes the proof of Theorem 3.2 for the space $I^n \setminus E$.

3.3 Proposition.

Let M^n be a connected, second countable manifold and let $\varphi : I^n \setminus E \to M$ be as in Theorem 2.2 . Then φ induces a continuous group embedding

(1) $\Phi_H : \mathcal{H}^\partial(I^n \setminus E)_\kappa \hookrightarrow \mathcal{H}^\partial(M)_\kappa$ and a (linear) homeomorphism

(2) $\Phi_M : \mathcal{M}_g^\partial(I^n \setminus E)_\omega \cong \{ \mu \in \mathcal{M}_g^\partial(M)_\omega \mid \mu(\varphi(\partial I^n \setminus E)) = o\}$

such that the diagram

$$(3) \qquad \begin{tikzcd} \mathcal{H}^\partial(I^n \setminus E) \times \mathcal{M}_g^\partial(I^n \setminus E) \arrow[r] \arrow[d, "\Phi_H \times \Phi_M"'] & \mathcal{M}_g^\partial(I^n \setminus E) \arrow[d, "\Phi_M"] \\ \mathcal{H}^\partial(M) \times \mathcal{M}_g^\partial(M) \arrow[r] & \mathcal{M}_g^\partial(M) \end{tikzcd}$$

is commutative.

Furthermore, let $\mu_o \in M_g^\partial(I^n \setminus E)$ be given and let $\nu_o = \Phi_M(\mu_o) = \varphi_* \mu_o$. Then Φ_H restricts continuously to

$$(4) \quad \Phi_H : H^\partial(I^n \setminus E, \mu_o\text{-e-reg})_\kappa \to H^\partial(M, \nu_o\text{-e-reg})_\kappa$$

$$(5) \quad \Phi_{H,c} : H_c^\partial(I^n \setminus E, \mu_o\text{-e-reg})_{\underrightarrow{\lim}} \to H_c^\partial(M, \nu_o\text{-e-reg})_{\underrightarrow{\lim}}$$

and Φ_M induces homeomorphisms

$$(6) \quad \Phi_M : M_g^\partial(I^n \setminus E, \mu_o\text{-e-reg})_{e\omega} \cong M_g^\partial(M, \nu_o\text{-e-reg})_{e\omega}$$

$$(7) \quad \Phi_{M,c} : M_{g,c}^\partial(I^n \setminus E, \mu_o\text{-e-reg})_{\underrightarrow{\lim}} \cong M_{g,c}^\partial(M, \nu_o\text{-e-reg})_{\underrightarrow{\lim}}$$

Proof.

The existence of Φ_H is a consequence of the fact that the map $\varphi : I^n \setminus E \to M$, being closed, is also an identification map.

The fact that the map $\Phi_M : \mu \mapsto \varphi_*\mu$ has a continuous inverse is a consequence of Lemma 1.5 . We omit the datails of this proof.

□

The next lemma is due to Oxtoby and Ulam (see [19]). It is contained in the remark following Fathi [9 , Proposition 2.1, p.p. 51-52] ; (it also follows from Berlanga and Epstein [3, Lemma 3]).

3.4 Lemma.

Let M be a second countable manifold and let $\mu \in M^\partial(M)$. If

A is a closed subset of M with empty interior, then there exists a homeomorphism $h \in H^{\partial}(M)$ such that $h_* \mu(A) = o$.

□

Remark : The set A of interest is $\varphi(\partial I^n \setminus E)$ (see 3.3(2)).

Let E be a totally disconnected, compact subset of ∂I^n as in Theorem 2.2, and let μ_o be a measure on $I^n \setminus E$ having the same sets of measure zero as Lebesgue measure on I^n.

Let $\mathbb{P}$ be a topological space and let

$$\mu,\nu : \mathbb{P} \to M_g^{\partial}(I^n \setminus E,\ \mu_o\text{-e-reg})_{e\omega}$$

be two continuous maps.

For each compact set K in $I^n \setminus E$ define

$$\mathbb{P}(K,\ \mu,\ \nu) = \mathbb{P}(K) = \{p \in \mathbb{P} \mid \mu_p \text{ is } K\text{-related to } \nu_p\} .$$

Since the relation of two measures being K-related is closed (see 1.6), then $\{\mathbb{P}(K) \mid K \subset I^n \setminus E \text{ compact}\}$ is a coherent family of spaces in

$$\mathbb{P}_c(\mu,\nu) = \mathbb{P}_c = \{p \in \mathbb{P} \mid \mu_p \text{ is compactly related to } \nu_p\} .$$

With these preliminaries we can state Lemma 3.5 .

3.5 Lemma.

Let $L \subset I^n \setminus E$ be a closed n-cell such that for each

$A \in \mathcal{C}((I^n \setminus E) \setminus L)$, $Cl_{I^n}\ A$ is a clean cube in I^n (in the sense of Lemma 2.5). Then there is a continuous map

$$f : \mathbb{P} \to \mathcal{H}_c^{\partial}(I^n \setminus E,\ \mu_o\text{-e-reg})_K$$

such that

(1) $f_{p*}\ \mu_p|_L = \nu_p|_L$;

$f_{p*}\mu_p\ (A) = \nu_p(A)$ for each $A \in \mathcal{C}((I^n \setminus E) \setminus L)$;

(2) $\{ f_p^{-1} \}_{p \in \mathbb{P}}$ is an equicontinuous family in $I^n \setminus L$;

(3) If $\mu_p = \nu_p$ then $f_p = \mathrm{Id}$;

(4) If μ_p is L-related to ν_p, then $f_{p*}\ \mu_p = \nu_p$;

(5) $f|_{\mathbb{P}_c} : \mathbb{P}_c \to \mathcal{H}_c^{\partial}(I^n \setminus E,\ \mu_o\text{-e-reg})$ is stratified (in the sense of Appendix One, statement (3)).

In order to prove this lemma, we need two preliminary results. The first is a slight variation of Fathi [9, Lemma 3.4, p. 54] and is a "compact" analogue of 3.5 above

3.6 Lemma.

Let $\mathbb{P}$ be a topological space and let ℓ be the standard Lebesgue measure in I^n. Let $L \subset I^n$ be a closed n-cell such that for each $A \in \mathcal{C}(I^n \setminus L)$, $Cl\ A$ is a clean cube in I^n.

Let μ' and ν' be two continuous maps from the parameter space $\mathbb{P}$ into the space $M_g^\partial(I^n, \ell\text{-e-reg})_\omega$ of ℓ-biregular probabilities on I^n.

Then there exists a continuous map $f' : \mathbb{P} \to H^\partial(I^n, \ell\text{-e-reg})_\kappa$ such that

(1) $f'_{p*}\, \mu'_p \big|_L = \nu'_p \big|_L$;

$f'_{p*}\, \mu'_p(A) = \nu'_p(A)$ for each $A \in C(I^n \setminus L)$;

(2) $\{(f'_p)^{-1} \big|(I^n \setminus L)\}_{p \in \mathbb{P}}$ is an equicontinuous family ;

(3) If $\mu'_p = \nu'_p$ then $f_p = \mathrm{Id}$;

(4) If μ'_p is L-related to ν'_p, then $f_{p*}\, \mu'_p = \nu'_p$.

<u>Proof</u>.

Use step 2 of the proof of Lemma 3.4 in Fathi [9] to equate the masses of L and the components of $\mathrm{Cl}(I^n \setminus L)$, then apply the full force of Lemma 3.4 to L.

$\square$

3.7 <u>Lemma</u>.

Let X be a locally compact Hausdorff space and let $\lambda \in M(X)$.

Let

$$L : I \to \{ B \subset X \mid B \text{ is Borel } \}$$

be a function of the unit interval into the set of Borel subsets of X such that

(1) For every $\tau_o \in I$ and every neighbourhood U_o of $\operatorname{Fr} L_{\tau_o}$ in X there exists $\delta > o$ such that $L_{\tau_o} \Delta L_\tau \subset U_o$ whenever $|\tau - \tau_o| < \delta$ ($L_{\tau_o} \Delta L_\tau$ is the symmetric difference $L_{\tau_o} \setminus L_\tau \cup L_\tau \setminus L_{\tau_o}$);

(2) $\lambda(\operatorname{Fr} L_\tau) = o$ for each $\tau \in I$.

Then $\rho : M(X, \ll \lambda)_\omega \times I \to M(X)_\omega$ such that $\rho(\mu,\tau) = \mu|_{L_\tau}$ is continuous, where

$$M(X, \ll \lambda) = \{ \mu \in M(X) \mid B \subset X \text{ Borel } \lambda(B) = o \Rightarrow \mu(B) = o \}.$$

<u>Proof</u>.

Let $f : X \to \mathbb{R}$ be a continuous function with compact support (not identically zero). We need to check that $(\mu,\tau) \mapsto \int f \, d\rho(\mu,\tau)$ is continuous, so assume X is compact.

Let $\varepsilon > o$ and let $(\mu_o,\tau_o) \in M(X, \ll\lambda) \times I$. Let $\mathcal{U}_o$ be a compact neighbourhood of $\operatorname{Fr} L_{\tau_o}$ with $\mu_o(\mathcal{U}_o) < \varepsilon/(2 \operatorname{supremum}_{x\in X} |f(x)|)$. Lemma 1.3 implies that there is a neighbourhood N_1 of μ_o in $M(X,\ll \lambda)$ such that $\mu(\mathcal{U}_o) < \varepsilon/(2 \operatorname{supremum}_{x\in X} |f(x)|)$ for each $\mu \in N_1$. Since $\lambda(\operatorname{Fr} L_{\tau_o}) = o$, Lemma 1.5 implies that $\mu \mapsto \int \chi_{L_{\tau_o}} f \, d\mu$ is continuous in $M(X, <<\lambda)$, where $\chi_{L_{\tau_o}}$ denotes the characteristic function of L_{τ_o}. Hence we can choose a neighbourhood N_2 of μ_o in $M(X,<<\lambda)$ such that $\left| \int \chi_{L_{\tau_o}} f \, d\mu - \int \chi_{L_{\tau_o}} f \, d\mu_o \right| < \varepsilon/2$ for each $\mu \in N_2$. Let $\delta > o$ be such that $L_{\tau_o} \Delta L_\tau \subset U_o$ for each $\tau \in I$ satisfying $|\tau - \tau_o| < \delta$ and let $N = N_1 \cap N_2$. Then, if $\mu \in N$ and $\tau \in < \tau_o - \delta, \tau_o + \delta > \cap I$,

$$
\begin{aligned}
& \left| \int f \, d\rho(\mu,\tau) - \int f \, d\rho(\mu_o,\tau_o) \right| \\
&= \left| \int \chi_{L_\tau} f \, d\mu - \int \chi_{L_{\tau_o}} f \, d\mu_o \right| \\
&\le \left| \int \chi_{L_\tau} f \, d\mu - \int \chi_{L_{\tau_o}} f \, d\mu \right| \\
&\qquad + \left| \int \chi_{L_{\tau_o}} f \, d\mu - \int \chi_{L_{\tau_o}} f \, d\mu_o \right| \\
&< \int \chi_{U_o} |f| \, d\mu + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon .
\end{aligned}
$$

□

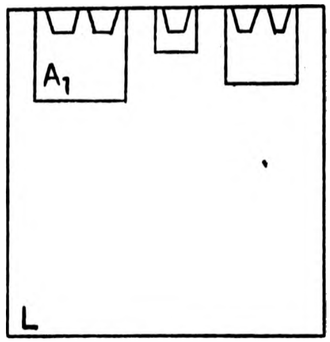
<u>Proof of Lemma</u> 3.5 .

Let $\Theta : I^{n-1} \times I \to I$ be a continuous function satisfying

(1) $L \subset L_\tau \subset \mathring{L}_{\tau'} \subset I^n \setminus E \subset L_1 = I^n$ for $o \le \tau < \tau' < 1$,

where $L_\tau = \{(x, s) \in I^n \mid o \le s \le \Theta(x, \tau)$ and $x \in I^{n-1}\}$;

(2) If $K \subset I^n \setminus E$ is compact then $K \subset L_\tau$ for some $\tau \in [o, 1 >$.



Let $A_1, \ldots, A_k, A_{k+1}, \ldots, A_q$ be the components of $(I^n \setminus E) \setminus L$ and assume that

(3) $\mu_o(A_i) < \infty$ for $1 \leq i \leq k$;

(4) $\mu_o(A_i) = \infty$ for $k+1 \leq i \leq q$.

Define, for each $1 \leq i \leq k+1$, $\alpha_i' : \mathbb{P} \times I \to \mathbb{R}$ such that

(5) $\alpha_i'(p, \tau) = \nu_p(A_i) - \mu_p(A_i \setminus L_\tau)$ for $1 \leq i \leq k$;

(6) $\alpha_{k+1}'(p, \tau) = \mu_p(L_\tau \cup \bigcup_{i=1}^{k} A_i) - \nu_p(L \cup \bigcup_{i=1}^{k} A_i)$.

Let $i = 1, 2, \ldots, k+1$ and let $p \in \mathbb{P}$. By the above lemma, and the fact that we are using the $e\omega$-topology, the function $\tau \mapsto \alpha_i'(p, \tau)$ is continuous. It is strictly increasing also and since $\alpha_i'(p, 1) > o$ there is a first parameter, say $\tau_i'(p)$, satisfying $\alpha_i'(p, \tau_i'(p)) \geq o$.

It is easy to see (by 3.7) that the function $p \mapsto \tau_i'(p)$ is continuous. Now let $s : \mathbb{P} \to I$ be such that

$$(7) \quad s(p) = \text{Max} \{ \tau_i'(p) \mid 1 \leq i \leq k+1 \} ;$$

let $\tau : P \to I$ be such that

$$(8) \quad \tau(p) = s(p) + (1 - s(p)) / 10 \; ;$$

and let $\alpha_i : \mathbb{P} \to \langle o, \infty \rangle \quad (1 \leq i \leq k+1)$ be such that

$$(9) \quad \alpha_i(p) = \alpha_i'(p, \tau(p)).$$

Observe that for each $s \in [o, 1 \rangle$ and each $p \in \mathbb{P}(L_s, \mu, \nu)$, $s(p) \leq s$ and so $\tau(p) \leq s + (1 - s) / 10 < 1$.

Also we have that $\alpha_i(p) > o$, for $1 \leq i \leq k+1$.

Define $\nu' : \mathbb{P} \to M(I^n \setminus E, \ll\mu_o)$ by the formula

$$(10) \quad \nu_p' = \sum_{i=1}^{k} \left[\frac{\alpha_i(p)}{\mu_p(A_i \cap L_{\tau(p)})} \right] \mu_p \Big|_{A_i \cap L_{\tau(p)}} +$$

$$+ \nu_p \Big|_L + \left[\frac{\alpha_{k+1}(p)}{\mu_p \left(\bigcup_{i=k+1}^{q} A_i \cap L_{\tau(p)} \right)} \right] \mu_p \Big|_{\bigcup_{i=k+1}^{q} A_i \cap L_{\tau(p)}} .$$

Lemma 3.7 implies that $p \mapsto \nu'_p$ is continuous. A calculation shows that $\nu'_p(L_{\tau(p)}) = \mu_p\,(L_{\tau(p)})$.

If μ_p and ν_p are L-related then $\nu'_p = \nu_p$ in $L_{\tau(p)}$.

Now we would like to construct a homeomorphism f_p supported in $L_{\tau(p)}$ depending continuosly on $p \in \mathbb{P}$ and such that $f_{p*}\,\mu_p = \nu_p$ in L.

We want to apply Lemma 3.6, but using the "moving" cell $L_{\tau(p)}$ as a base space. The following device will fix it : Define a continuous function

$$g : I \longrightarrow H^{\partial}(I^{n-1} \times [\,o,\,2\,],\ \ell\text{-e-reg})$$

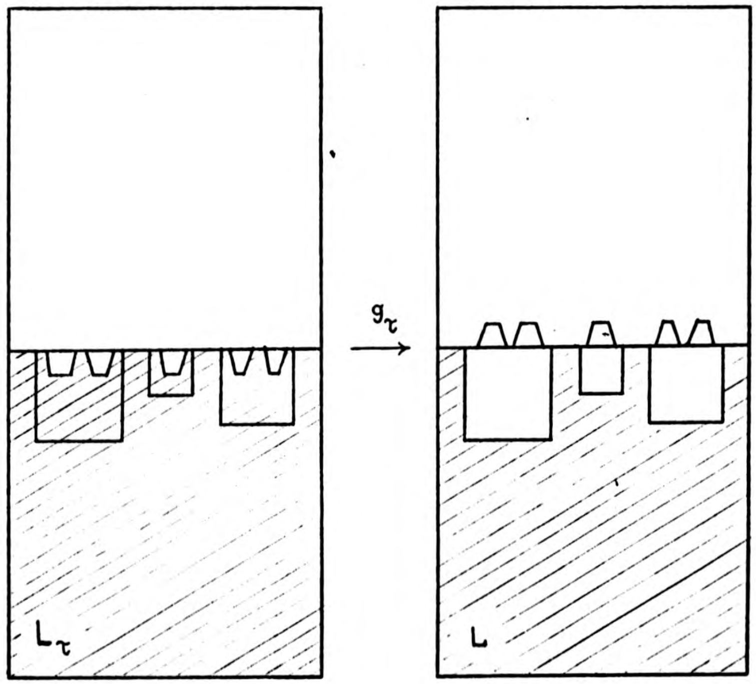
(where ℓ denotes Lebesgue measure on $I^n \times [\,o,\,2]$) satisfying

(11) supp $g_\tau \subset \mathrm{Cl}(I^{n-1} \times [\,o,\,2\,] \setminus L)$ for $\tau \in I$,

(12) $g_\tau(L_\tau) = I^n$ for $\tau \in I$,

(13) $g_\tau(\mathrm{Cl}\,A_i \cap L_\tau) = \mathrm{Cl}\,A_i$ for $\tau \in I$ and $1 \le i \le q$,

(14) $\{g_\tau\}_{\tau \in I}$ and $\{g_\tau^{-1}\}_{\tau \in I}$ are equicontinuous families (this is immediate for both families are compact).



Now we can apply Lemma 3.6 to

$$(15) \quad p \mapsto \frac{1}{\mu_p(L_{\tau(p)})} \, g_{\tau(p)*} \, \mu_p \Big|_{I^n}$$

and

$$(16) \quad p \mapsto \frac{1}{\mu_p(L_{\tau(p)})} \, g_{\tau(p)*} \nu_p' \Big|_{I^n}$$

to get a continuous $f' : \mathbb{P} \to H^{\partial}(I^n, \ell\text{-e-reg})$ satisfying certain properties.

Define

$$f : \mathbb{P} \to H_c^{\partial}(I^n \setminus E, \mu_o\text{-e-reg})$$

by the formula

$$f_p = g_{\tau(p)}^{-1} \circ f_p' \circ g_{\tau(p)} \qquad \text{for each } p \in \mathbb{P} .$$

This is the map demanded in the statement of the lemma.

□

Now we apply Lemma 3.5 to obtain, in slightly modified terms, Theorem 3.2 for the space $I^n \setminus E$.

The proof is a straightforward (although somewhat careful) induction over an increasing sequence of n-cells.

Recall that if μ_p and ν_p are two measures depending on $p \in \mathbb{P}$, then $\mathbb{P}_c$ denotes the set of parameters p for which μ_p is compactly related to ν_p.

3.8 <u>Lemma.</u>

Let $\mathbb{P}$ be a topological space and let

$\mu, \nu : \mathbb{P} \to M_g^\partial(I^n \setminus E, \mu_o\text{-e-reg})_{e\omega}$ be two continuous maps, where $\mu_o \in M_g^\partial(I^n \setminus E)$ is a measure having the same sets of measure zero as standard Lebesgue measure in I^n. Then there is a continuous map

$$h : \mathbb{P} \to H^\partial(I^n \setminus E)_\kappa$$

such that

$$h_{p*}\,\mu_p = \nu_p \quad \text{for each} \quad p \in \mathbb{P}.$$

Furthermore, $h(\mathbb{P}_c) \subset H_c^\partial(I^n \setminus E)$ and

$$h : \mathbb{P}_c \to H_c^\partial(I^n \setminus E)$$

is stratified.

Proof.

Let $K_o = \emptyset \subsetneq \overset{\circ}{K}_1 \subset K_1 \subset \overset{\circ}{K}_2 \subset K_2 \subset \ldots \subset I^n \setminus E$ be a sequence of closed n-cells such that $\cup K_i = I^n \setminus E$ and $Cl_{I^n} A$ is a clean cube in I^n for each $A \in C((I^n \setminus E) \setminus K_i)$ and each $i \in \mathbb{N}$.

Let $L_o = K_o$, $h_p^o = Id$ for each $p \in \mathbb{P}$ and let $L_1 = K_1$. Therefore, by the above lemma we can find $h^1 : \mathbb{P} \to H_c^\partial(I^n \setminus E, \mu_o\text{-e-reg})_\kappa$ such that

$$\text{(3-1)} \quad h^1_{p*}\,\mu_p\big|_{L_1} = \nu_p\big|_{L_1} \quad \text{for } p \in \mathbb{P}, \text{ and}$$

$$h^1_{p*}\,\mu_p(A) = \nu_p(A) \quad \text{for } p \in \mathbb{P} \text{ and } A \in C((I^n \setminus E) \setminus L_1);$$

(4-1) $\{(h_p^1)^{-1}\}_{p\in\mathbb{P}}$ is an equicontinuous family in $I^n \setminus \mathring{L}_1$;

(5-1) If $p \in \mathbb{P}(K_0, \mu, \nu)$ (i.e. $\mu_p = \nu_p$) then $h_p^1 = \mathrm{Id}$;

(6-1) If $p \in \mathbb{P}(K_1, \mu, \nu)$ then $h_{p*}^1 \mu_p = \nu_p$;

(7-1) $h^1|_{\mathbb{P}_c}$ is stratified.

Now we can find, by (4-1) and (7-1), an integer $n_2 > 1$ so large that

(a-1) If $A \in C((I^n \setminus E) \setminus K_{n_2})$ and $p \in \mathbb{P}$ then diam $A < 1/2$ and diam $(h_p^1)^{-1}(A) < 1/2$;

(b-1) If $p \in \mathbb{P}$ and μ_p is K_2-related to ν_p then $h_{p*}^1 \mu_p$ is K_{n_2}-related to ν_p .

Let $L_2 = K_{n_2}$. Applying Lemma 3.5 to the closure of each $A \in C(I^n \setminus L_1)$ we can get an $h^2 : \mathbb{P} \to H_c^\partial(I^n \setminus E, \mu_0\text{-e-reg})_\kappa$ such that

(2-2) supp $h_p^2 \subset (I^n \setminus E) \setminus \mathring{L}_1$ for $p \in \mathbb{P}$;

(3-2) $h_{p*}^1 \mu_p|_{L_2} = h_{p*}^2 \nu_p|_{L_2}$ for $p \in \mathbb{P}$, and

$h_{p*}^1 \mu_p(A) = h_{p*}^2 \nu_p(A)$ for $p \in \mathbb{P}$ and $A \in C((I^n \setminus E) \setminus L_2)$;

(4-2) $\{(h_p^2)^{-1}\}_{p \in \mathbb{P}}$ is an equicontinuous family in $I^n \setminus \mathring{L}_2$;

(5-2) If $p \in \mathbb{P}(K_1, \mu, \nu)$ then $h_p^2 = \mathrm{Id}$;

(6-2) If $p \in \mathbb{P}(K_2, \mu, \nu)$ then $h^1_{p*}\mu_p = h^2_{p*}\nu_p$;

(7-2) $h^2|_{\mathbb{P}_c}$ is stratified.

Hence, by induction, we can construct an increasing sequence $\{L_i\}_{i \in \mathbb{N}}$ of closed cells, and a sequence of continuous maps $\{h^i : \mathbb{P} \to H_c^\partial(I^n \setminus E, \mu_o\text{-e-reg})\}_{i \in \mathbb{N}}$ such that, for each $i \in \mathbb{N} \setminus \{o\}$,

(1-i) $K_i \subset L_i$;

(2-i) $\mathrm{supp}\ h_p^i \subset (I^n \setminus E) \setminus \mathring{L}_{i-1}$ for all $p \in \mathbb{P}$,

(3-i) $h_p^i \circ h_p^{i-2} \circ \ldots \circ h^1_{p*}\mu_p|_{L_i} = h_p^{i-1} \circ h_p^{i-3} \circ \ldots \circ h^o_{p*}\nu_p|_{L_i}$,

and

$$h_p^i \circ h_p^{i-2} \circ \ldots \circ h^1_{p*}\mu_p(A) = h_p^{i-1} \circ h_p^{i-3} \circ \ldots \circ h^o_{p*}\nu_p(A)$$

for all $A \in C((I^n \setminus E) \setminus L_i)$ and i odd;

$$h_p^{i-1} \circ h_p^{i-3} \circ \ldots \circ h^1_{p*}\mu_p|_{L_i} = h_p^i \circ h_p^{i-2} \circ \ldots \circ h^o_{p*}\nu_p|_{L_i} ,$$

and

$$h_p^{i-1} \circ h_p^{i-3} \circ \ldots \circ h^1_{p*}\mu_p(A) = h_p^i \circ h_p^{i-2} \circ \ldots \circ h^o_{p*}\nu_p(A)$$

for all $A \in C((I^n \setminus E) \setminus L_i)$ and i even ;

(4-i) $\{(h_p^i)^{-1}\}_{p \in \mathbb{P}}$ is an equicontinuous family in $I^n \setminus \overset{\circ}{L}_i$;

(5-i) If $p \in \mathbb{P}(K_{i-1}, \mu, \nu)$ then $h_p^i = \mathrm{Id}$;

(6-i) If $p \in \mathbb{P}(K_i, \mu, \nu)$ then

$h_p^i \circ h_p^{i-2} \circ \ldots \circ h^1_{p*}\, \mu_p = h_p^{i-1} \circ h_p^{i-3} \circ \ldots \circ h^o_{p*}\, \nu_p$ for i odd,

and $h_p^{i-1} \circ h_p^{i-3} \circ \ldots \circ h^1_{p*}\, \mu_p = h_p^i \circ h_p^{i-2} \circ \ldots \circ h^o_{p*}\, \nu_p$ for i even.

(7-i) $h^i|_{\mathbb{P}_c(\mu,\nu)}$ is stratified ;

(8-i) diam $A < 1/2^i$,

diam $(h_p^i \circ h_p^{i-2} \circ \ldots \circ h_p^{i \bmod(2)})^{-1}(A) < 1/2^i$

for all $A \in C((I^n \setminus E) \setminus L_{i+1})$ and all $p \in \mathbb{P}$.

By (2-(i + 2)) and (8-i), h_p^{i+2} is a $(1/2^i)$-homeomorphism, hence

$$\lim_{i\to\infty} H^i_{\text{even},p} = \lim_{i\to\infty} h_p^{2i} \circ h_p^{2i-2} \circ \ldots \circ h_p^o = H_{\text{even},\,p}$$

and

$$\lim_{i\to\infty} H^i_{\text{odd},p} = \lim_{i\to\infty} h_p^{2i+1} \circ h_p^{2i-1} \circ \ldots \circ h_p^1 = H_{\text{odd},\,p}$$

exist and are continuous as functions in $\mathbb{P}$.

Also, by (8-2i) and (8-(2i + 1)),

$$d((H^{i}_{even,p})^{-1}, (H^{i+1}_{even, p})^{-1})$$

$$= d((H^{i}_{even, p})^{-1}, (H^{i}_{even, p})^{-1} \circ (h_p^{2i+2})^{-1})$$

$$\leq 1/2^{2i}$$

and

$$d((H^{i}_{odd,p})^{-1}, (H^{i+1}_{odd,p})^{-1}) \leq 1/2^{2i+1} \quad ,$$

hence $H_{odd,p}$ and $H_{even,p}$ are in $H^{\partial}(I^n \setminus E)$. Furthermore, $H^{-1}_{even} \circ H_{odd} = h$ is stratified in P_c and $h_{p*}\mu_p = \nu_p$ for all $p \in \mathbf{P}$.

This concludes the proof.

□

Proof of Theorem 3.2

Without loss of generality assume that M is connected.

Let $\mu_o \in M^{\partial}_g(M)$ and consider the map $\pi_o : H^{\partial}(M, \mu_o\text{-e-reg})_{\kappa} \to M^{\partial}_g(M, \mu_o\text{-e-reg})_{e\omega}$ such that $\pi_o(h) = h_* \mu_o$ for each $h \in H^{\partial}(M, \mu_o\text{-e-reg})$.

Let $\varphi' : I^n \setminus E \to M$ be a continuous surjection as in Theorem 2.2. Then, by Lemma 3.4 we can assume that $\varphi'(\partial I^n \setminus E)$ has zero μ_o-measure. Hence, by Proposition 3.3 there exists a measure ν'_o in $I^n \setminus E$ with

$\Phi'_M(\nu'_0) = \varphi'_* \nu'_0 = \mu_0$. It is easy to construct a good measure in $I^n \setminus E$, say ν_0, having the same sets of measure zero as standard Lebesgue measure in $I^n \setminus E$, the same total mass as ν'_0, and such that $\alpha(\nu_0) = \alpha(\nu'_0)$. Therefore, by Theorem 1.15, there exists a homeomorphism f of $I^n \setminus E$ in $H_0^\partial(I^n \setminus E)$ such that $f_* \nu_0 = \nu'_0$. If we let $\varphi = \varphi' \circ f$ then $\Phi_M(\nu_0) = \varphi_* \nu_0 = \mu_0$. Using Lemma 3.8, we can find a continuous section σ' of

$\pi'_0 : H^\partial(I^n \setminus E, \nu_0\text{-e-reg})_\kappa \to M_g^\partial(I^n \setminus E, \nu_0\text{-e-reg})_{e\omega}$ with

$h' \mapsto h'_* \nu_0$ for each $h' \in H^\partial(I^n \setminus E, \nu_0\text{-e-reg})$. Finally, it is easy to check (by Proposition 3.3) that the composite

$$\begin{tikzcd}
M_g^\partial(M, \mu_0\text{-e-reg})_{e\omega} \arrow[r, "\Phi_M^{-1}"] \arrow[ddr, "\sigma"'] & M_g^\partial(I^n \setminus E, \nu_0\text{-e-reg})_{e\omega} \arrow[d, "\sigma'"] \\
& H^\partial(I^n \setminus E, \nu_0\text{-e-reg}) \arrow[d, "\Phi_H"] \\
& H^\partial(M, \mu_0\text{-e-reg})_\kappa
\end{tikzcd}$$

is a section σ of $\pi_0 : H^\partial(M, \mu_0\text{-e-reg}) \to M_g^\partial(M, \mu_0\text{-e-reg})$ such that σ restricts continuously to

$$\sigma_c : M_{g,c}^\partial(M, \mu_0\text{-e-reg})_{\lim} \to H_c^\partial(M, \mu_0\text{-e-reg})_{\lim} .$$

This is because Φ_M^{-1}, σ' and Φ_H are all stratified when they are appropriately restricted (see Proposition 3.3 and Lemma 3.8). $\square$

Now we want to draw two corollaries from Theorem 3.2 . They state that some groups of measure preserving homeomorphisms are (strong) deformation retracts of their corresponding groups of e-biregular homeomorphisms.

First we need some definitions.

3.9 Definitions

Let X be a locally compact, locally connected, second countable, Hausdorff space, and let

$\mu_0 \in \mathcal{M}_g(X)$. Define

$$\mathcal{H}(X, \mu_0) = \{ h \in \mathcal{H}(X) \mid h_* \mu_0 = \mu_0 \} ,$$

$$\mathcal{H}(K, X, \mu_0) = \mathcal{H}(K, X) \cap \mathcal{H}(X, \mu_0) \text{ for each } K \subset X \text{ compact},$$

$$\mathcal{H}_c(X, \mu_0) = \mathcal{H}_c(X) \cap \mathcal{H}(X, \mu_0).$$

Observe that Proposition 1.4 implies that $\mathcal{H}(X, \mu_0)$ is closed in $\mathcal{H}(X)_\kappa$ and Proposition 1.7 implies that $\mathcal{H}_c(X, \mu_0)$ is closed in $\mathcal{H}_c(X)_{\underrightarrow{\lim}}$. Also, by Appendix One, the topology that $\mathcal{H}_c(X, \mu_0)$ gets as a subspace of $\mathcal{H}_c(X)_{\underrightarrow{\lim}}$ is the same as the topology coinduced by the coherent family $\{\mathcal{H}(K, X, \mu_0)_\kappa \mid K \subset X \text{ compact}\}$.

If $X = M$ is a manifold, let

$$\mathcal{H}^\partial(M, \mu_0) = \mathcal{H}(M, \mu_0) \cap \mathcal{H}^\partial(M)$$

$$\mathcal{H}^\partial(K, M, \mu_0) = \mathcal{H}(K, M, \mu_0) \cap \mathcal{H}^\partial(M)$$

$$H^{\partial}_{c}(M, \mu_0) = H_c(M, \mu_0) \cap H^{\partial}(M).$$

Similarly, $H^{\partial}(M, \mu_0)$ is closed in $H(M)_{\kappa}$ (hence in $H(M, \mu_0)_{\kappa}$) and $H^{\partial}_{c}(M, \mu_0)$ is closed in $H_c(M)_{\underset{\rightarrow}{\lim}}$ (hence in $H_c(M, \mu_0)_{\underset{\rightarrow}{\lim}}$).

3.10 <u>Proposition</u>.

Let M be a second countable manifold, and let $\mu_0 \in M^{\partial}_{g}(M)$. Then

(1) $H^{\partial}(M, \mu_0\text{-e-reg})_{\kappa}$ is homeomorphic to the product $H^{\partial}(M, \mu_0)_{\kappa} \times M^{\partial}_{g}(M, \mu_0\text{-e-reg})_{e\omega}$;

(2) $H^{\partial}(M, \mu_0)_{\kappa}$ is a (strong) deformation retract of $H^{\partial}(M, \mu_0\text{-e-reg})_{\kappa}$,

and

(3) $H^{\partial}_{c}(M, \mu_0\text{-e-reg})_{\underset{\rightarrow}{\lim}}$ is homeomorphic to the space $[\, H^{\partial}_{c}(M,\mu_0) \times M^{\partial}_{g,c}(M, \mu_0\text{-e-reg})\,]_{\underset{\rightarrow}{\lim}}$;

(4) $H^{\partial}_{c}(M,\mu_0)_{\underset{\rightarrow}{\lim}}$ is a (strong) deformation retract of $H^{\partial}_{c}(M, \mu_0\text{-e-reg})_{\underset{\rightarrow}{\lim}}$.

<u>Proof</u>.

By Theorem 3.2 the surjective map $\pi_0 : H^{\partial}(M, \mu_0\text{-e-reg})_{\kappa} \rightarrow M^{\partial}_{g}(M, \mu_0\text{-e-reg})_{e\omega}$ has a continuous section σ.

A homeomorphism of $\mathcal{H}^{\partial}(M, \mu_o\text{-e-reg})_\kappa$ onto $\mathcal{H}^{\partial}(M,\mu_o)_\kappa \times \mathcal{M}_g^{\partial}(M, \mu_o\text{-e-reg})_{e\omega}$ is defined by

$$F(h) = (F_1(h), F_2(h)) = ([\sigma(\pi_o(h))]^{-1} \circ h, \pi_o(h))$$

for $h \in \mathcal{H}^{\partial}(M, \mu_o\text{-e-reg})$,

and its inverse F^{-1} is given by the formula

$$F^{-1}(h,\mu) = ([\sigma(\mu)] \circ h)$$

for $(h,\mu) \in \mathcal{H}^{\partial}(M,\mu_o) \times \mathcal{M}_g^{\partial}(M, \mu_o\text{-e-reg})$.

This proves (1). The function

$$H_1 : \mathcal{M}_g^{\partial}(M, \mu_o\text{-e-reg})_{e\omega} \times I \to \mathcal{M}_g^{\partial}(M, \mu_o\text{-e-reg})_{e\omega}$$

such that

$$(\mu, \tau) \mapsto (1 - \tau)\mu + \tau\mu_o$$

is well defined and continuous (c f. Lemma 1.16). Therefore $\{\mu_o\}$ is a retract by deformation of $\mathcal{M}_g^{\partial}(M, \mu_o\text{-e-reg})$. Also, we have that $F(h) = (h,\mu_o)$ for each $h \in \mathcal{H}^{\partial}(M,\mu_o)$; hence it follows that the homotopy

$$H_2 : \mathcal{H}^{\partial}(M, \mu_o\text{-e-reg})_\kappa \times I \to \mathcal{H}^{\partial}(M, \mu_o\text{-e-reg})_\kappa$$

$$(h, \tau) \mapsto F^{-1}(F_1(h), H_1(F_2(h), \tau))$$

defines $H^{\partial}(M,\mu_o)$ as a (strong) deformation retract of $H^{\partial}(M,\mu_o\text{-e-reg})_{\kappa}$.

To prove (3) it is enough to observe that F(resp. F^{-1}) restricted to $H_c^{\partial}(M, \mu_o\text{-e-reg})$ (resp. $H_c^{\partial}(M,\mu_o) \times M_{g,c}^{\partial}(M, \mu_o\text{-e-reg})$) is stratified.

Finally, by the fact that the composition of stratified functions is stratified, it follows that H_2' restricts to

$$H_{2,c} : H_c^{\partial}(M, \mu_o\text{-e-reg}) \times I \to H_c^{\partial}(M, \mu_o\text{-e-reg})$$

and $H_{2,c}$ is stratified. Also,

$$[\ H_c^{\partial}(M, \mu_o\text{-e-reg})_{\underset{\rightarrow}{\lim}}\ ] \times I \quad \approx$$

$$[\ H_c^{\partial}(M, \mu_o\text{-e-reg}) \times I\ ]_{\underset{\rightarrow}{\lim}}$$

for I is compact.

Since $H_c^{\partial}(M,\mu_o)_{\underset{\rightarrow}{\lim}}$ is a closed subspace in $H_c^{\partial}(M, \mu_o\text{-e-reg})_{\underset{\rightarrow}{\lim}}$, statement (4) now follows.

□

3.11 <u>Corollary</u>

Let M be a second countable manifold, and let $\mu_o \in M_g^{\partial}(M)$. Then

(1) $H(M, \mu_o\text{-e-reg})_{\kappa} \cong H(M,\mu_o)_{\kappa} \times M_g^{\partial}(M, \mu_o\text{-e-reg})_{e\omega}$;

(2) $\mathcal{H}(M, \mu_o)_K$ is a (strong) deformation retract of $\mathcal{H}(M, \mu_o\text{-e-reg})_K$,

and

(3) $\mathcal{H}_c(M, \mu_o\text{-e-reg})_{\underset{\rightarrow}{\lim}} \cong [\mathcal{H}_c(M, \mu_o) \times \mathcal{M}^{\partial}_{g,c}(M, \mu_o\text{-e-reg})]_{\underset{\rightarrow}{\lim}}$;

(4) $\mathcal{H}_c(M, \mu_o)_{\underset{\rightarrow}{\lim}}$ is a (strong) deformation retract of $\mathcal{H}_c(M, \mu_o\text{-e-reg})_{\underset{\rightarrow}{\lim}}$.

Proof.

Since a section of $\pi_o : \mathcal{H}^{\partial}(M, \mu_o\text{-e-reg}) \to \mathcal{M}^{\partial}_g(M, \mu_o\text{-e-reg})$ is also a section of $\pi_o : \mathcal{H}(M, \mu_o\text{-e-reg}) \to \mathcal{M}^{\partial}_g(M, \mu_o\text{-e-reg})$, we can prove this corollary in the same way as Lemma 3.10 .

□

§4. The local contractibility of $H_c(M, \mu_o)$ and its equivalence under weak homotopy to the group of all compactly supported homeomorphisms of M.

Let M be a second countable manifold and let μ_o be a ∂-good measure on M. The main results of this section are: Firstly, that the group of compactly supported, measure preserving homeomorphisms of M is locally contractible in the $\underset{\rightarrow}{\lim}$-topology. Secondly, that the inclusion $H_c(M,\mu_o)_{\underset{\rightarrow}{\lim}} \hookrightarrow H_c(M)_{\underset{\rightarrow}{\lim}}$ is a weak homotopy equivalence.

We start Section 4 by considering the Whitney topology on the group of homeomorphisms of M. The relationship between the Whitney topology and the $\underset{\rightarrow}{\lim}$-topology on the group of compactly supported homeomorphisms is established by saying that both topologies have the same family of compact subsets. This observation is of some importance to further investigations.

In Fathi [9], the Černavskii, Edwards-Kirby Theorem on the deformation of spaces of embeddings is adapted to the case of spaces of μ_o-biregular embeddings. This and the results of Section 3 are the main tools to prove that $\mathcal{H}_c(M, \mu_o)$ is locally contractible.

Finally, by 3.11, we know that the inclusion $\mathcal{H}_c(M, \mu_o) \hookrightarrow \mathcal{H}_c(M, \mu_o\text{-e-reg})$ is a homotopy equivalence. Hence it remains to prove that $\mathcal{H}_c(M, \mu_o\text{-e-reg}) \hookrightarrow \mathcal{H}_c(M)$ is a weak homotopy equivalence.

Although the spaces we are considering are not metrizable in general, the methods of Eilenberg-Wilder (see [8]) apply to our situation.

4.1 Definition

Let X be a locally compact, locally connected, Hausdorff space. The fine or Whitney topology m on $\mathcal{H}(X)$ is the topology having for a basis the sets

$$[\ \{K_i\}_{i\in\Lambda}\ ,\ \{U_i\}_{i\in\Lambda}\ ] = \{h \in \mathcal{H}(X) \mid h(K_i) \subset U_i \text{ for all } i\in\Lambda\},$$

where $\{K_i\}_i$ is a locally finite family of compact sets and $\{U_i\}_i$ an (arbitrary) open family in X.

With this topology $\mathcal{H}(X)_m$ becomes a topological group. Observe that

(a) $\mathcal{H}(K, X)_m = \mathcal{H}(K, X)_K$ for $K \subset X$ compact;

(b) $\mathcal{H}_c(X)$ is a closed (nowhere dense) set in $\mathcal{H}(X)_m$;

(c) The inclusions $\mathcal{H}_c(X)_{\underrightarrow{\lim}} \hookrightarrow \mathcal{H}(X)_m \hookrightarrow \mathcal{H}(X)_K$ are continuous.

If X has a metric d, then a base for the Whitney topology is given by the sets

$$N(h_0, \varepsilon) = \{h \in H(X) \mid d(h_0(x), h(x)) < \varepsilon(x) \quad \text{all} \quad x \in X\},$$

where $h_0 \in H(X)$ and $\varepsilon : X \to \langle 0, \infty \rangle$ is continuous.

4.2 <u>Definitions</u>

Let X be a topological Hausdorff space. We say that X is a k-<u>space</u> (or <u>compactly generated space</u>) if the following property is satisfied:

> Let C be a subset of X. Then C is closed in X if and only if its intersection with each compact subset of X is closed.

Let X be a Hausdorff space. Then the <u>associated compactly generated space</u> is the space $k(X)$ defined as follows:

(1) $k(X)$ and X have the same underlying set ;

(2) A subset C of $k(X)$ is defined to be closed whenever its intersection with every compact subset of X is closed.

Some properties of compactly generated spaces are listed in the second appendix to this thesis.

4.3 <u>Lemma.</u>

Let X be a locally compact, locally connected, second countable, Hausdorff space. Let $\mathbb{K} \subset H_c(X)_m$ be compact. Then there is a compact subset K in X such that every element in $\mathbb{K}$ has support in K.

Proof.

We argue by contradiction: Assume that a compact subset $\mathbb{K}$ in $\mathcal{H}_c(X)_m$ is given such that for every compact K in X there is an $h \in \mathbb{K}$ with $\operatorname{supp} h \not\subset K$.

It is easy to construct, by induction, a sequence $\{K_i\}_{i=1}^{\infty}$ of compact subsets of X and a sequence $\{h_i\}_{i=1}^{\infty}$ of homeomorphisms in $\mathbb{K}$ such that, for each $i \in \mathbb{N} \setminus \{o\}$,

(1) $K_i \subset \mathring{K}_{i+1}$,

(2) $\bigcup_{j=1}^{\infty} K_j = X$,

(3) $\operatorname{supp} h_i \not\subset K_i$;

(4) $\operatorname{supp} h_i \subset K_{i+1}$.

We want to show that $\{h_i\}_{i=1}^{\infty}$ is a closed and not compact subset of $\mathbb{K}$. For this purpose, let $h_o \in \mathcal{H}_c(X) \setminus \{h_i\}_{i=1}^{\infty}$ and choose, for each $i \in \mathbb{N}$, open sets $N_{1,i}$ and $N_{2,i}$ in $\mathcal{H}_c(X)$ such that

(5) $h_i \in N_{1,i} \cap N_{2,i}$;

(6) $h_j \notin N_{1,i}$ for $o \leq j < i$;

(7) $h_j \notin N_{2,i}$ for $i < j$.

Therefore, $N_{1,o} \cap N_{2,o}$ is a neighbourhood of h_o contained in $\mathcal{H}_c(X) \setminus \{h_i\}_{i=1}^{\infty}$, proving that $\{h_i\}_{i=1}^{\infty}$ is closed. Also,

$\{N_{1,i} \cap N_{2,i}\}_{i=1}^{\infty}$ is an open cover of $\{h_i\}_{i=1}^{\infty}$ which has no finite subcover.

□

4.4 Corollary.

Let X be a locally compact, locally connected, second countable, Hausdorff space and let $\mathcal{U} \subset \mathcal{H}_c(X)_m$ be an open subspace. Let $\mathcal{U}_{\underset{\rightarrow}{\lim}}$ denote the space obtained from (the underlying set of) $\mathcal{U}$ with the topology coinduced by the (coherent) family

$$\{\mathcal{H}(K, X)_K \cap \mathcal{U} \mid K \subset X \text{ compact}\}.$$

Let $\mathcal{U}_s$ be the space obtained from $\mathcal{U}$ with the subspace topology from $\mathcal{H}_c(X)_{\underset{\rightarrow}{\lim}}$. Finally, let $k\mathcal{U}$ be the associated compactly generated space of $\mathcal{U}$. Then

$$k\,\mathcal{U} = \mathcal{U}_{\underset{\rightarrow}{\lim}} = \mathcal{U}_s$$

and $\mathcal{U}_{\underset{\rightarrow}{\lim}}$ is open in

$$\mathcal{H}_c(X)_{\underset{\rightarrow}{\lim}} = k(\mathcal{H}_c(X)_m).$$

Proof.

Since the inclusion $\mathcal{H}_c(X)_{\underset{\rightarrow}{\lim}} \hookrightarrow \mathcal{H}_c(X)_m$ is continuous, $\mathcal{U}_s$ is open in $\mathcal{H}_c(X)_{\underset{\rightarrow}{\lim}}$. Also, $\mathcal{U}_s = \mathcal{U}_{\underset{\rightarrow}{\lim}}$ holds by Appendix Two (10), (and the fact that X is σ-compact).

By Lemma 4.3 and Appendix One (5), $\mathcal{H}_c(X)_{\underset{\rightarrow}{\lim}}$ and $\mathcal{H}_c(X)_m$ have the same compact sets, so $\mathcal{H}_c(X)_{\underrightarrow{\lim}} = k(\mathcal{H}_c(X)_m)$. Finally, by Appendix Two (9), $k\mathcal{U}$ is just $\mathcal{U}_s$.

□

4.5 Remark

We are mostly interested in the implication :
"If $\mathcal{U} \subset \mathcal{H}_c(X)_m$ is open then $\mathcal{U}_{\underrightarrow{\lim}} \subset \mathcal{H}_c(X)_{\underrightarrow{\lim}}$ is open" but we had to go via k-spaces to obtain it.

An interesting corollary of this approach is that we know now that the inclusion $\mathcal{H}_c(X)_{\underrightarrow{\lim}} \hookrightarrow k(\mathcal{H}(X)_m)$ is a closed embedding, giving us us an alternative way of viewing the $\underrightarrow{\lim}$-topology in $\mathcal{H}_c(X)$.

Before we can state the Černavskii-Edwards-Kirby Theorem for spaces of biregular embeddings we need some definitions.

4.6 Definitions.

Let M be a (second countable) manifold and let $\mu_o \in M_g^{\partial}(M)$.

Let A be a subset of M. By a proper embedding ι of A into M we mean an injective (continuous) map $\iota : A \hookrightarrow M$ such that ι is a homeomorphism of A onto $\iota(A)$ and $\iota^{-1}(\partial M) = A \cap \partial M$.

Denote by $I(A, M)$ the space of proper embeddings of A into M.

If $\iota \in I(A, M)$ and A is a Borel subset of M, we can define a measure $\iota^*\mu_o$ on A such that $\iota^*\mu_o(B) = \mu_o(\iota(B))$ for each Borel subset $B \subset A$. We say that a proper embedding $\iota : A \hookrightarrow M$ is biregular (with respect to μ_o), if $\iota^*\mu_o$ and $\mu_o|A$ have the same sets of measure zero. Denote by $I(A, M; \ll\mu_o\ll)$ the set of all proper biregular embeddings of A into M.

Suppose B is a subset of M. We define

$$I(A, B, M) = \{ \iota \in I(A, M) \mid \iota|_{B \cap A} = \mathrm{Id} \},$$

$$I(A, B, M; \ll\mu_o\ll) = I(A, B, M) \cap I(A, M, \ll\mu_o\ll).$$

All spaces of proper embeddings will be endowed with the compact open topology.

Suppose X is a space with subsets Q and S. A <u>deformation</u> <u>of</u> Q <u>into</u> S is a continuous map $\phi : Q \times I \to X$ such that $\phi \mid Q \times \{0\} = \mathrm{Id}_Q$ and $\phi(Q \times \{1\}) \subset S$. If $T \subset X$ and $\phi(Q \times I) \subset T$, we say that ϕ <u>takes</u> <u>place</u> <u>in</u> T.

Let P be a subset of $I(A, M)$ and W a subset of A. A deformation $\phi : P \times I \to I(A, M)$ of P is <u>modulo</u> W if $\phi(\iota, t)|_W = \iota|_W$ for all $\iota \in P$ and $t \in I$.

The following theorem, where no measures intervene, is due to Černavskii (see [5]).

A much more readable and elegant aproach, again without mention to measures, is due to Edwards-Kirby (see [7]).

The version which follows appears in the paper of Fathi (see [9]). A.Fathi gives the credit for this result to M.Rogalski.

4.7 Theorem

Let M be a (second countable) manifold and let $\mu_0 \in M_g^{\partial}(M)$. Let C and U be subsets of M such that C is compact and U is a neighbourhood of C. Given any neighbourhood N of the inclusion $\eta : U \hookrightarrow M$ in $I(U, M; \langle\langle\mu_0\langle\langle)_K$, there is a neighbourhood P of η in $I(U, M; \langle\langle\mu_0\langle\langle)_K$ and a deformation $\phi : P \times I \to N$ into $I(U, C, M; \langle\langle\mu_0\langle\langle)$ such that

(1) ϕ is modulo the complement of a compact neighbourhood of C in $\mathring{U}$;

(2) $\phi(\eta, t) = \eta$ for all $t \in I$;

(3) $\phi|[P \cap I(U, \partial M, M; \langle\langle\mu_0\langle\langle)] \times I$ takes place in $I(U, \partial M, M; \langle\langle\mu_0\langle\langle)$.

Furthermore, suppose in addition to the above hypothesis that a closed set D in M (resp. ∂M) and a neighbourhood V of D in M (resp. ∂M) are given. Then ϕ can be chosen so that the deformation $\phi \mid [P \cap I(U, V, M; \langle\langle\mu_0\langle\langle)] \times I$ takes place in $I(U, D, M; \langle\langle\mu_0\langle\langle)$.

□

Recall that a space X is <u>locally contractible</u> if for each $x \in X$ there is a neighbourhood U of x such that U is deformable into $\{x\}$ by a deformation fixing x .

This condition implies that every neighbourhood V of a point $x \in X$ contains a neighbourhood U of x deformable to $\{x\}$ in V.

4.8 Proposition.

Let M be a (second countable) manifold and let $\mu_o \in M_g^\partial(M)$. Then $\mathcal{H}_c(M, \mu_o\text{-e-reg})_{\underset{\rightarrow}{\lim}}$ and $\mathcal{H}_c^\partial(M, \mu_o\text{-e-reg})_{\underset{\rightarrow}{\lim}}$ are locally contractible.

Proof. (c f. Edwards and Kirby [7, Corollary 6.2, p.78]).

It suffices to prove that the identity has a contractible neighbourhood.

Fix a metric d on M. If $U \subset M$ is compact and $\delta > o$, let $N(\eta, \delta)$ denote the (basic) neighbourhood of the inclusion $\eta: U \hookrightarrow M$ in $I(U, M; \ll\mu_o\ll)$ given by the set
$\{ \iota \in I(U, M; \ll\mu_o\ll) \mid d(\iota(x), x) < \delta \text{ for all } x \in U \}$.

Let $\{(U_i, C_i) \mid i \in \mathbb{N}\}$ be a countable collection of pairs of compact subsets of M such that for each $i \in \mathbb{N}$, U_i is a neighbourhood of C_i, $M = \bigcup_{i \in \mathbb{N}} \overset{\circ}{C}_i$ and $U_i \cap U_j \neq \emptyset$ only if $|i-j| \leq 1$.

It follows from Theorem 4.7 (letting $U = U_{2i}$, $C = C_{2i}$, $V = C_{2i-1} \cup C_{2i+1}$ and $D = Cl(U_{2i} \setminus C_{2i})$) that there is a sequence $\{\delta_{2i}\}$ of positive numbers such that if P_{2i} is defined to be the neighbourhood $N(\eta, \delta_{2i})$ of $\eta: U_{2i} \hookrightarrow M$ in $I(U_{2i}, M; \ll\mu_o\ll)_K$ then there is a deformation

$\phi_{2i} : P_{2i} \times I \to I(U_{2i}, M; \ll\mu_o\ll)_K$ of P_{2i} into $I(U_{2i}, C_{2i}, M; \ll\mu_o\ll)$ such that ϕ_{2i} deforms $P_{2i} \cap I(U_{2i}, C_{2i-1} \cup C_{2i+1}, M; \ll\mu_o\ll)$ into $\{\eta\}$ and ϕ_{2i} is modulo $Fr_M U_{2i}$.

Likewise, there is a sequence $\{\delta_{2i-1}\}$ of positive numbers such that if P_{2i-1} is defined to be the neighbourhood $N(\eta, \delta_{2i-1})$ of $\eta : U_{2i} \hookrightarrow M$ in $I(U_{2i-1}, M; \langle\langle\mu_o\langle\langle)_\kappa$, then there is a deformation

$\phi_{2i-1} : P_{2i-1} \times I \to I(U_{2i-1}, M; \langle\langle\mu_o\langle\langle)_\kappa$ of P_{2i} into

$I(U_{2i-1}, C_{2i-1}, M; \langle\langle\mu_o\langle\langle)$ such that ϕ_{2i-1} takes place in

$N(\eta, \mathrm{MIN}\{\delta_{2i-2}, \delta_{2i}\})$ and ϕ_{2i-1} is modulo $Fr_M U_{2i-1}$.

Let $\delta: M \to \langle o, \infty\rangle$ be continuous and such that $\delta(U_i) < \delta_i$ for each $i \in \mathbb{N}$ and let $\mathcal{U}_\kappa$ be the set

$\mathcal{U} = \{h \in \mathcal{H}(M, \mu_o\text{-e-reg}) \mid d(h(x), x) < \delta(x)$ for all $x \in M\}$
endowed with the compact open topology.

Define the continuous function ϕ: $\mathcal{U}_\kappa \times I \to \mathcal{H}(M, \mu_o\text{-e-reg})_\kappa$ by the formula

$$\phi(h, t) = \begin{cases} \phi_{2i-1}(h \mid U_{2i-1}, 2t) & \text{on } U_{2i-1} \text{ for } t \in [o, 1/2], \\ h & \text{on } M \setminus \bigcup_{i \in \mathbb{N}} U_{2i-1}. \end{cases}$$

$$\phi(h, t) = \begin{cases} \phi_{2i}(\phi(h, 1/2) \mid U_{2i}, 2t-1) & \text{on } U_{2i} \text{ for } t \in [1/2, 1], \\ \phi(h, 1/2) & \text{on } M \setminus \bigcup_{i \in \mathbb{N}} U_{2i}. \end{cases}$$

Furthermore, ϕ restricts to

$\phi_c : \mathcal{U}_c \times I \to \mathcal{H}_c(M, \mu_o\text{-e-reg})$ where $\mathcal{U}_c = \mathcal{U} \cap \mathcal{H}_c(M, \mu_o\text{-e-reg})$ and it is stratified, (because $\phi([\mathcal{U} \cap \mathcal{H}(\bigcup_{j=0}^{2i-1} C_j, M, \mu_o\text{-e-reg})] \times I)$ is contained in $\mathcal{H}(\bigcup_{j=0}^{2i} C_j, M, \mu_o\text{-e-reg})$). Hence

$\phi_c : [\mathcal{U}_c \times I]_{\underrightarrow{\lim}} \to \mathcal{H}_c(M, \mu_o\text{-e-reg})_{\underrightarrow{\lim}}$ is continuous. Also $[\mathcal{U}_c \times I]_{\underrightarrow{\lim}} = [\mathcal{U}_c]_{\underrightarrow{\lim}} \times I$ (by Appendix Two (7)) and $[\mathcal{U}_c]_{\underrightarrow{\lim}}$ is an open subspace in $\mathcal{H}_c(M, \mu_o\text{-e-reg})_{\underrightarrow{\lim}}$ (by 4.4). This implies that $\phi_c : [\mathcal{U}_c]_{\underrightarrow{\lim}} \times I \to \mathcal{H}_c(M, \mu_o\text{-e-reg})_{\underrightarrow{\lim}}$ is a deformation of $\mathcal{U}_c$ into $\{\eta\}$.

By theorem 4.7, $\phi(\{\eta\} \times I) = \{\eta\}$ and $\phi| [\mathcal{U}_c \cap \mathcal{H}_c^{\partial}(M, \mu_o\text{-e-reg})] \times I$ takes place in $\mathcal{H}_c^{\partial}(M, \mu_o\text{-e-reg})$. Hence the conclusion of the proposition follows.

□

Recall that a space X is <u>semi-locally simply connected</u> if for each $x \in X$ there is a neighbourhood U of x in X such that the homomorphism $\pi(U, x) \to \pi(X, x)$ between the fundamental groups of U and X (at x) induced by the inclusion $U \hookrightarrow X$ is trivial. Obviously, a locally contractible space X is semi-locally simply connected.

4.9 <u>Theorem.</u>

Let M be a second countable manifold and let $\mu_o \in \mathcal{M}_g^{\partial}(M)$. Then $\mathcal{H}_c(M,\mu_o)_{\underrightarrow{\lim}}$ and $\mathcal{H}_c^{\partial}(M,\mu_o)_{\underrightarrow{\lim}}$ are (locally path connected and) locally contractible.

Furthermore, there exists a (Whitney) neighbourhood $\mathcal{U}$ of the identity in $\mathcal{H}_c(M,\mu_o)_m$ such that $\mathcal{U}_{\underrightarrow{\lim}}$ is deformed into {Id} by a contraction ϕ fixing the identity and such that $\phi \mid [\mathcal{H}_c^{\partial}(M,\mu_o)_{\underrightarrow{\lim}} \cap \mathcal{U}_{\underrightarrow{\lim}}] \times I$ takes place in $\mathcal{H}_c^{\partial}(M,\mu_o)$. Hence $\mathcal{H}_c(M,\mu_o)_m$ and $\mathcal{H}_c^{\partial}(M,\mu_o)_m$ are locally path connected and semi-locally simply connected in the Whitney topology.

Proof.

Let x be a point in a space X. Then X is connected im kleinen at x if and only if each open neighbourhood V of x contains an open neighbourhood U of x such that any pair of points in U lie in some connected subset K of V. It is not difficult to show that if X is connected im kleinen at each point, then X is locally connected. X is locally path connected if the subsets K in the above definition can be taken to be continuous images of the unit interval.

Let $\mathcal{U}_c$ be a Whitney neighbourhood of the identity in $\mathcal{H}_c(M, \mu_o\text{-e-reg})$, contractible in the $\underrightarrow{\lim}$-topology, such as in Proposition 4.8 . If $\mathcal{U} = \mathcal{U}_c \cap \mathcal{H}_c(M, \mu_o)$, then there is a contraction (by 3.10 and 3.11) $\phi : \mathcal{U}_{\underrightarrow{\lim}} \times I \to \mathcal{H}_c(M,\mu_o)_{\underrightarrow{\lim}}$ with the desired properties, proving that $\mathcal{H}_c(M, \mu_o)_{\underrightarrow{\lim}}$ and $\mathcal{H}_c^{\partial}(M, \mu_o)_{\underrightarrow{\lim}}$ are locally contractible, hence connected im kleinen at each point and also locally path connected. Since any function γ from a compact space P (e.g. P = I or P = I × I) into $\mathcal{H}_c(X, \mu_o)$ is continuous in the $\underrightarrow{\lim}$-topology if and only if it is continuous in the Whitney topology, the last assertion of the theorem follows.

□

4.10 In [8], S. Eilenberg and R.L. Wilder investigate "the properties of uniformly locally connected subsets of a metric separable space with particular reference to the relation between the set and its closure".

If a subset A contained in the separable metric space X is uniformly locally j-connected for $j = 0, 1, \ldots, q$; then the theorem of Eilenberg and Wilder states, in particular, that A and $Cl_X A$ have the same homotopy and homology groups for the dimensions $0, 1, \ldots, q$.

The Černavskii, Edwards-Kirby Theorem implies a "stratified" local contractibility for the space of compactly supported biregular homeomorphisms of a manifold. Hence, it gives a corresponding form of "stratified" uniform local j-connectedness for each $j \in \mathbb{N}$.

Also, by Fathi [9 , Lemma 4.7] , it is easily seen that $\mathcal{H}_c(M, \mu_o\text{-e-reg})_{\underrightarrow{\lim}}$ is dense in $\mathcal{H}_c(M)_{\underrightarrow{\lim}}$.

Although $\mathcal{H}_c(M)_{\underrightarrow{\lim}}$ is not metrizable when M is non-compact, we can adapt the methods of Eilenberg and Wilder in order to prove the following result.

4.11 <u>Theorem.</u>

Let M be a connected, second countable manifold, and let $\mu_o \in M_g^\partial(M)$. Then the inclusions $\mathcal{H}_c(M, \mu_o)_{\underrightarrow{\lim}} \hookrightarrow \mathcal{H}_c(M)_{\underrightarrow{\lim}}$ and $\mathcal{H}_c^\partial(M, \mu_o)_{\underrightarrow{\lim}} \hookrightarrow \mathcal{H}_c^\partial(M)_{\underrightarrow{\lim}}$ are weak homotopy equivalences.

Proof.

The proof for both cases is the same. Therefore, we restrict ourselves to consider the inclusion $H_c(M, \mu_o) \hookrightarrow H_c(M)$.

By 3.11(4), the inclusion $H_c(M, \mu_o)_{\underrightarrow{\lim}} \hookrightarrow H_c(M, \mu_o\text{-e-reg})_{\underrightarrow{\lim}}$ is a homotopy equivalence. Hence, we need only to prove that the inclusion $H_c(M, \mu_o\text{-e-reg})_{\underrightarrow{\lim}} \hookrightarrow H_c(M)_{\underrightarrow{\lim}}$ is a weak homotopy equivalence.

We divide the proof into 3 steps.

Step 1.

Let d be a fixed metric on M.

Define the right invariant metric d^* on $H_c(M)$ by the formula

$$d^*(f, g) = \underset{x \in X}{\text{supremum}}\ d(f(x), g(x)) ,$$

for each two homeomorphisms of M with compact support.

Observe that the inclusion $H_c(M)_{\underrightarrow{\lim}} \hookrightarrow H_c(M)_{d^*}$ is continuous.

By Berlanga and Epstein [3 , Lemma 7], we can construct an increasing sequence $\emptyset \neq K_o \subset \mathring{K}_1 \subset K_1 \subset \mathring{K}_2 \subset K_2 \subset \ldots \subset M$ of relative cells whose union is M and such that $\mu_o(\mathrm{Fr}\ K_i) = o$ for each $i \in \mathbb{N}$.

The following properties are satisfied:

(1) $H(K_i, M, \mu_o\text{-e-reg})$ is dense in $H(K_i, M)$ for each $i \in \mathbb{N}$.

Therefore $H_c(M, \mu_o\text{-e-reg})_{\underset{\rightarrow}{\lim}}$ is dense in $H_c(M)_{\underset{\rightarrow}{\lim}}$.

(2) Let $i \in \mathbb{N}$ and let $\varepsilon > 0$.

Then there exists a $\delta > 0$ such that each continuous map $\gamma^j : \partial I^{j+1} \to H(K_i, M, \mu_o\text{-e-reg})$ with diameter $\left[\gamma^j(\partial I^{j+1})\right] < \delta$ is extended continuously to some $\overline{\gamma^j} : I^{j+1} \to H(K_{i+1}, M, \mu_o\text{-e-reg})$ with diameter $\left[\overline{\gamma^j}(I^{j+1})\right] < \varepsilon$.

Remarks. The first part of (1) is essentially Fathi [9 , Lemma 4.7, p.61]. Property (2) is a consequence of Theorem 4.7 above and the right invariance of the metric d^* .

We conclude this step with the following definition.

Definition.

Let $\gamma : \partial I^{j+1} \to H_c(K_i, M, \mu_o\text{-e-reg})$ be a continuous map. Define a number $b(\gamma)$ as follows: $b(\gamma) = \infty$ if γ does not have an extension $\bar{\gamma} : I^{j+1} \to H_c(K_{i+1}, M, \mu_o\text{-e-reg})$; and

$b(\gamma) = \text{g.l.b.}\ \{\text{diameter } \bar{\gamma}(I^{j+1}) \mid \bar{\gamma} : I^{j+1} \to H(K_{i+1}, M, \mu_o\text{-e-reg})$ is a continuous extension of $\gamma\}$ otherwise.

Step 2 (c f. Eilenberg and Wilder [8, Theorem 1]).

Let B be a closed subset of a compact metric space Z such that the topological dimension of $Z \setminus B$ is finite.

Let $f : B \to H_c(M)_{\underset{\rightarrow}{\lim}}$ be a continuous map. Then there is an open subset $U \supset B$ and a continuous extension $f' : U \to H_c(M)_{\underset{\rightarrow}{\lim}}$ of f

such that $f'(U \setminus B) \subset \mathcal{H}_c(M, \mu_o\text{-e-reg})$.

Proof.

Since B is compact, then $f(B) \subset \mathcal{H}(K_i, M)$ for some $i \in \mathbb{N}$. So assume, without loss of generality that $i = o$.

Suppose that $\dim(Z \setminus B) \leq q$. Then, according to a theorem of Kuratowski [16 , Theorem 2], $\mathcal{H}(K_o, M)$ can be isometrically embedded in a metric separable space Y such that

(1) $\mathcal{H}(K_o, M)$ is a closed subset of Y.

(2) $Y \setminus \mathcal{H}(K_o, M) = P^q$ is an infinite polyhedron of dimension less or equal q .

(3) The map $f : B \to \mathcal{H}(K_o, M)$ has a continuous extension $\bar{f} : Z \to Y$ such that $\bar{f}(Z \setminus B) \subset P^q$.

In view of (3) it is therefore sufficient to prove that

(4) There is an open set V such that $\mathcal{H}(K_o, M) \subset V \subset Y$ and a continuous function $\rho : V \to \mathcal{H}(K_q, M)$ such that $\rho(h) = h$ for each $h \in \mathcal{H}(K_o, M)$ and $\rho(V \setminus \mathcal{H}(K_o, M)) \subset \mathcal{H}(K_q, M, \mu_o\text{-e-reg})$.

We proceed to construct the set V and the mapping ρ in exactly the same way as in Eilenberg and Wilder [8 , Theorem 1, p.615] :

Let P^j denote the j-dimensional skeleton of P^q ; i.e., the subpolyhedron of P^q consisting only of the simplices of dimension less or equal j. To prove (4) it is sufficient to prove that for $j = 0, 1, \ldots, q$;

(5) There is an open set V_j such that $H(K_o, M) \subset V_j \subset Y$ and a continuous function $\rho_j: H(K_o, M) \cup (V_j \cap P^j) \to H(K_j, M)$ such that $\rho_j(h) = h$ for each $h \in H(K_o, M)$ and $\rho_j(V_j \cap P^j) \subset H(K_j, M, \mu_o\text{-e-reg})$.

Hence we may proceed by induction. For $j = o$, P^o is the set of all vertices of P^q. Since $H(K_o, M, \mu_o\text{-e-reg})$ is dense in $H(K_o, M)$, we can find for every vertex $x_\alpha \in P^o$ a point $h_\alpha \in H(K_o, M, \mu_o\text{-e-reg})$ such that $d^*(x_\alpha, h_\alpha) < 2\, d^*(x_\alpha, H(K_o, M))$. Defining $V_o = Y$ and $\rho_o(h) = h$ for $h \in H(K_o, M)$, $\rho_o(x_\alpha) = h_\alpha$ for $x_\alpha \in P^o$, we verify (5) for $j = o$.

Suppose that V_j and ρ_j of (5) are given for some $j < q$. Let $\Delta_1^{j+1}, \Delta_2^{j+1}, \ldots,$ be the $(j + 1)$-dimensional simplices contained in $V_j \cap P^q$. Denote by S_α^j the boundary sphere of Δ_α^{j+1}. The mapping $\rho_j : S_\alpha^j \to H(K_j, M, \mu_o\text{-e-reg})$ is therefore defined. Denote this partial mapping by $\gamma_\alpha : S_\alpha^j \to H(K_j, M, \mu_o\text{-e-reg})$. If $b(\gamma_\alpha) < \infty$ then we can find a continuous extension $\overline{\gamma}_\alpha : \Delta_\alpha^{j+1} \to H(K_{j+1}, M, \mu_o\text{-e-reg})$ of γ_α such that diameter $\left[\overline{\gamma}_\alpha(\Delta_\alpha^{j+1})\right] < 2\, b(\gamma_\alpha)$. Now if the subsequence $\{\Delta_{\alpha_s}^{j+1}\}$ converges to a point $h \in H(K_o, M)$ then diameter $\left[\rho_j(S_{\alpha_s}^j)\right] \to o$ and diameter $\left[\gamma_{\alpha_s}(S_{\alpha_s}^j)\right] \to o$. By Step 1 (2), we must have $b(\gamma_{\alpha_s}) \to o$ and therefore $\overline{\gamma}_{\alpha_s}(\Delta_{\alpha_s}^{j+1})$ converges to h.

Consequently there is an open set V_{j+1} such that

$H(K_o, M) \subset V_{j+1} \subset Y$ and $b(\gamma_\alpha)$ is finite whenever $\Delta_\alpha^{j+1} \cap V_{j+1} \neq \emptyset$.
Taking $\rho_{j+1}(x) = \rho_j(x)$ for $x \in H(K_o, M) \cup (V_{j+1} \cap P^j)$;
$\rho_{j+1}(x) = \overline{\gamma}_\alpha(x)$ for $x \in V_{j+1} \cap \Delta_\alpha^{j+1}$, we verify (5) for $j + 1$.

This concludes the proof. of Step 2.

$\square$

The following discussion is taken, with minor modification, from Eilenberg and Wilder [8 , Theorem 2].

Step 3

Let B be a closed subset of a compact space such that $\dim(Z \setminus B)$ is finite. Let $f : Z \to H_c(M)_{\underset{\to}{\lim}}$ be continuous. Then there is a continuous function $f^* : Z \to H_c(M)_{\underset{\to}{\lim}}$ satisfying the following properties:

(1) f is homotopic to f^* ;

(2) $f^*(Z \setminus B) \subset H_c(M, \mu_o\text{-e-reg})$;

(3) $f^*(z) = f(z)$ for $z \in B$.

Proof.

Consider the product space $Z^* = Z \times I$ and the closed subspace $B^* = Z \times \{o\} \cup B \times I$.

Clearly $Z^* \setminus B^*$ is of finite dimension. Define the mapping $g : B^* \to H_c(M)_{\underset{\to}{\lim}}$ as follows:

$$g(z, o) = f(z) \quad \text{for} \quad z \in Z ;$$

$$g(z, t) = f(z) \quad \text{for} \quad (z, t) \in B \times I.$$

By Step 2, we can find an open set U such that $B^* \subset U \subset Z^*$ and an extension $g^* : U \to H_c(M)_{\underset{\to}{\lim}}$ of g^* such that $g^*(U \setminus B^*) \subset H_c(M, \mu_o\text{-e-reg})$. For sufficiently small $t_o \in I$ we then have that $Z \times [\, o, t_o \,] \subset U$. Taking $f^*(z) = g^*(z, t_o)$ we get the desired function.

□

It now follows, from Step 3, that the inclusion $H_c(M, \mu_o\text{-e-reg})_{\underset{\to}{\lim}} \hookrightarrow H_c(M)_{\underset{\to}{\lim}}$ is a weak homotopy equivalence. This concludes the proof of Theorem 4.11 .

$$g(z, o) = f(z) \quad \text{for} \quad z \in Z ;$$

$$g(z, t) = f(z) \quad \text{for} \quad (z, t) \in B \times I.$$

By Step 2, we can find an open set U such that $B^* \subset U \subset Z^*$ and an extension $g^* : U \to \mathcal{H}_c(M)_{\underrightarrow{\lim}}$ of g^* such that $g^*(U \setminus B^*) \subset \mathcal{H}_c(M, \mu_o\text{-e-reg})$. For sufficiently small $t_o \in I$ we then have that $Z \times [\, o, t_o \,] \subset U$. Taking $f^*(z) = g^*(z, t_o)$ we get the desired function.

□

It now follows, from Step 3, that the inclusion $\mathcal{H}_c(M, \mu_o\text{-e-reg})_{\underrightarrow{\lim}} \hookrightarrow \mathcal{H}_c(M)_{\underrightarrow{\lim}}$ is a weak homotopy equivalence. This concludes the proof of Theorem 4.11 .

§5 The extension of measure preserving isotopies of a compact subset in a manifold.

We apply here the Kirby-Edwards theorem (Theorem 4.7) and a parametrized version of the von Neumann-Oxtoby-Ulam theorem to show that, under certain circumstances, measure preserving perturbations and measure preserving isotopies of a compact subset in a manifold can be extended to measure preserving homeomorphisms and measure preserving ambient isotopies respectively. Theorem 5.6 is used in section 8 below (see the *Introduction*).

5.1 Proposition

Let M be a second countable manifold and let $\mu_o \in \mathcal{M}_g^{\partial}(M)$. Let C and U be subsets of M such that C is compact and U is a neighbourhood of C. Then there is a neighbourhood P of the inclusion $\eta : U \hookrightarrow M$ in $I(U, M; \ll\mu_o\ll)$ and a continuous map

$$P \to H_C(M, \mu_o\text{- e-reg})$$

$$\iota \mapsto \bar{\iota}$$

such that

(1) $\bar{\iota}|_C = \iota|_C$;

(2) There is a compact neighbourhood F of C in $\mathring{U}$, independent of ι, such that $\operatorname{supp} \bar{\iota} \subset F$;

(3) $\bar{\eta} = Id_M$;

(4) If ι fixes $U \cap \partial M$ pointwise, then $\bar{\iota}$ fixes ∂M pointwise.

Furthermore, suppose in addition to the above hypothesis that a closed set D in M(resp. ∂M) and a neighbourhood V of D in M (resp. ∂M) are given. Then the correspondence $\iota \mapsto \bar{\iota}$ can be chosen so that, for each ι which fixes $U \cap V$ pointwise, its extension $\bar{\iota}$ fixes D pointwise.

Proof.

Let ϕ be the deformation given in Theorem 4.7 . Then

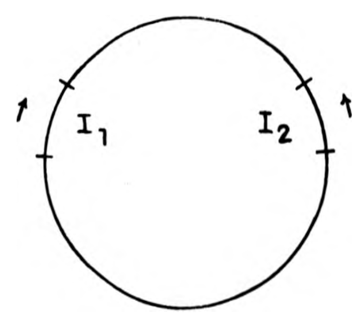
$$\iota \circ \phi(\iota, 1)^{-1} : \phi(\iota, 1)(U) \to M$$

is equal to ι on C and is the identity outside $\phi(\iota, 1)(F)$, hence it can be extended by the identity to a homeomorphism of M.

This concludes the proof.

□

The analogue of Theorem 4.7 is false for measure preserving embeddings. We give an example which contradicts the above result: Take two small intervals in the unit circle S^1, say I_1 and I_2, and push them one towards the other using rotations. This small push cannot be extended to a measure preserving homeomorphism of S^1



We now give analogues of 5.1 for the measure preserving case.

5.2 <u>Proposition</u>.

Let M be a connected, second countable manifold, and let $\mu_o \in \mathcal{M}_g^{\partial}(M)$. Let C and U be subsets of M such that C is compact and U is a neighbourhood of C.

If $M \setminus C$ is connected, then there is a neighbourhood P_{μ_o} of the inclusion $\eta : U \hookrightarrow M$ in $I(U, M; \mu_o)$ and a compact neighbourhood F of C in M (not necessarily in $\mathring{U}$) such that for each $\iota \in P_{\mu_o}$ there is a measure preserving homeomorphism $\tilde{\iota} \in H_c(M, \mu_o)$ with the following properties:

(1) $\tilde{\iota}$ depends continuously on ι ;

(2) $\tilde{\iota}|C = \iota|C$;

(3) supp $\tilde{\iota} \subset F$;

(4) $\tilde{\eta} = Id_M$;

(5) If ι fixes $U \cap \partial M$ pointwise, then $\tilde{\iota}$ fixes ∂M pointwise;

(6) Furthermore, suppose in addition to the above hypothesis that a closed set D in ∂M and a neighbourhood V of D in ∂M are given. Then the correspondence $\iota \mapsto \tilde{\iota}$ can be chosen so that, for each ι which fixes $U \cap V$ pointwise, its extension $\tilde{\iota}$ fixes D pointwise.

Proof.

Let C^+ be a compact neighbourhood of C in M such that $C^+ \subset \mathring{U}$. By applying Proposition 5.1 to the pair (U, C^+) we get a neighbourhood P of μ_o-biregular embeddings $\iota : U \hookrightarrow M$, a compact set $F^- \subset M$, and a continuous function $\iota \mapsto \bar{\iota}$ on P satisfying certain properties.

and U is a neighbourhood of C.

If $M \setminus C$ is connected, then there is a neighbourhood P_{μ_0} of the inclusion $\eta : U \hookrightarrow M$ in $I(U, M; \mu_0)$ and a compact neighbourhood F of C in M (not necessarily in $\mathring{U}$) such that for each $\iota \in P_{\mu_0}$ there is a measure preserving homeomorphism $\tilde{\iota} \in H_c(M, \mu_0)$ with the following properties:

(1) $\tilde{\iota}$ depends continuously on ι ;

(2) $\tilde{\iota}|C = \iota|C$;

(3) $\operatorname{supp} \tilde{\iota} \subset F$;

(4) $\tilde{\eta} = Id_M$;

(5) If ι fixes $U \cap \partial M$ pointwise, then $\tilde{\iota}$ fixes ∂M pointwise;

(6) Furthermore, suppose in addition to the above hypothesis that a closed set D in ∂M and a neighbourhood V of D in ∂M are given. Then the correspondence $\iota \mapsto \tilde{\iota}$ can be chosen so that, for each ι which fixes $U \cap V$ pointwise, its extension $\tilde{\iota}$ fixes D pointwise.

<u>Proof.</u>

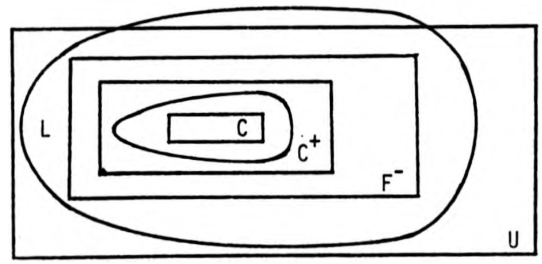
Let C^+ be a compact neighbourhood of C in M such that $C^+ \subset \mathring{U}$. By applying Proposition 5.1 to the pair (U, C^+) we get a neighbourhood P of μ_0-biregular embeddings $\iota : U \hookrightarrow M$, a compact set $F^- \subset M$, and a continuous function $\iota \mapsto \bar{\iota}$ on P satisfying certain properties.

Lema 7 in Berlanga and Epstein [3] implies that there is a relative cell L contained in M such that

$$L \cap C = \emptyset$$

$$F^- \setminus \mathring{C}^+ \subset L$$

$$\mu_o(Fr\, L) = o$$



It is not difficult to verify that $b = \mu_o(\bar{\iota}(L))$ is independent of $\iota \in P_{\mu_o}$, where $P_{\mu_o} = P \cap I(U, M, \mu_o)$.

Define $M_g^B(L, \mu_o|_L\text{-bireg})$ to be the space of all measures $\nu \in M_g(L)$ with total mass equal to b and having the same sets of measure zero as $\mu_o|_L$ (in particular, $(\partial M \cap L) \cup F_r L$ is a set of zero ν-measure).

Define the continuous function

$$P_{\mu_o} \to M_g^B(L, \mu_o|_L\text{-bireg})$$

$$\iota \mapsto (\tau^{-1}{}_*\mu_o) \,|\, L$$

where $M_g^B(L,\mu_o|_L\text{-bireg})$ is endowed with the weak topology induced by the family of continuous functions $f : L \to \mathbb{R}$.

The fact that L is a relative cell implies that Theorem 3.2 is valid for L (see the diagram in the proof of Proposition 1 in [3]). That is, we can find a continuous map

$$\sigma : M_g^B(L,\mu_o|_L\text{-bireg}) \to H^\partial(L, M, \mu_o\text{-e-reg})$$

Lema 7 in Berlanga and Epstein [3] implies that there is a relative cell L contained in M such that

$L \cap C = \emptyset$

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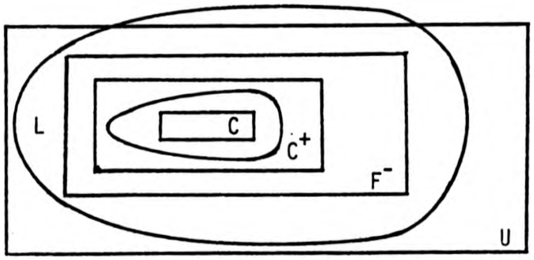


It is not difficult to verify that $b = \mu_o(\bar{\iota}(L))$ is independent of $\iota \in P_{\mu_o}$, where $P_{\mu_o} = P \cap I(U, M, \mu_o)$.

Define $M_g^B(L, \mu_o|_L\text{-bireg})$ to be the space of all measures $\nu \in M_g(L)$ with total mass equal to b and having the same sets of measure zero as $\mu_o|_L$ (in particular, $(\partial M \cap L) \cup F_r L$ is a set of zero ν-measure).

Define the continuous function

$$P_{\mu_o} \to M_g^B(L, \mu_o|_L\text{-bireg})$$

$$\iota \mapsto (\tau^{-1}{}_*\mu_o)\,|\,L$$

where $M_g^B(L,\mu_o|_L\text{-bireg})$ is endowed with the weak topology induced by the family of continuous functions $f : L \to \mathbb{R}$.

The fact that L is a relative cell implies that Theorem 3.2 is valid for L (see the diagram in the proof of Proposition 1 in [3]). That is, we can find a continuous map

$$\sigma : M_g^B(L,\mu_o|_L\text{-bireg}) \to H^\partial(L, M, \mu_o\text{-e-reg})$$

such that

$$\sigma(\nu) * \mu_o|_L = \nu$$

Let $F = C^+ \cup L$. Then

$$\mathcal{P}_{\mu_o} \to \mathcal{H}_C(M, \mu_o)$$

$$\iota \mapsto \bar{\iota} \circ \sigma(\bar{\iota}^{-1} {}_* \mu_o|_L)$$

is the required μ_o-measure preserving extension function.

□

Now we want to generalize Proposition 5.2 to the case in which the set $\mathcal{C}(M \setminus C)$ of connected components of $M \setminus C$ is finite and $\mu(M) < \infty$.

First consider the following example: Let S^2 be the two-dimensional sphere, and let $S^1 \subset S^2$ be the equator. Then a reflection of S^2 along S^1 fixes S^1 pointwise and sends one hemisphere into the other.

The next remarks tell that such homeomorphisms are always far from the identity, and that under some mild hypothesis they do not exist at all.

5.3 <u>Remarks</u>

Let M be a manifold, and let C be a compact subset of M such that $\mathcal{C}(M \setminus C)$ is finite. Then there exists a (compact-open) neighbourhood $\mathcal{N}$ of the identity in $\mathcal{H}(M)$ such that if $f \in \mathcal{N}$ fixes C pointwise, then $f(A) = A$ for each $A \in \mathcal{C}(M \setminus C)$. More precisely, the set

S of homeomorphisms $f : M \to M$ such that $f|_C = Id_C$ and $f(A) = A$ for each $A \in C(M \setminus C)$ is both open and closed (in the compact open topology) when considered as a subset of the space of all homeomorphisms f of M fixing C pointwise.

Furthermore, suppose that C is such that $\mathrm{Fr}\, A_1 \neq \mathrm{Fr}\, A_2$ for each two distinct components A_1 and A_2 of $M \setminus C$. Then the set S of homeomorphisms defined above is equal to the full group of homeomorphisms of M fixing C pointwise. As an example of this situation, let M be a connected manifold, and let C be a locally flat codimension zero submanifold of M. Then the frontiers of any two distinct components of $M \setminus C$ are disjoint and non-empty.

5.4

Let M be a connected, second countable manifold and let $\mu_o \in M_g^{\partial}(M)$ with $\mu_o(M) < \infty$. Let $C \subset M$ be a compact subset such that $C(M \setminus C)$ is finite, and let $U \subset M$ be a neighbourhood of C.

Let C^+ be a compact neighbourhood of C in M such that $C^+ \subset \mathring{U}$. By applying Proposition 5.1 to the pair (U, C^+) we get a neighbourhood P of μ_o-biregular embeddings $\iota : U \hookrightarrow M$, and a continuous function $\iota \mapsto \bar{\iota}$ on P satisfying certain properties.

Observe that the above remarks imply that $\bar{\iota}(A)$ $(A \in C(M \setminus C)$ is independent of the particular extension of $\iota|_{C^+}$ to M.

We now state the generalization of Proposition 5.2

5.5 Proposition.

Consider the situation of 5.4 and let $P_{\mu_o} = P \cap I(U, M; \mu_o)$. There exists a compact neighbourhood F of C in M such that for each $\iota \in P_{\mu_o}$ satisfying

$$\mu_o(\bar{\iota}(A)) = \mu_o(A) \qquad \text{for each } A \in C(M \setminus C)$$

there is a measure preserving homeomorphism $\tilde{\iota} \in H_c(M, \mu_o)$ for which properties (1) to (6) of Proposition 5.2 hold.

The proof is the same as that of 5.2

$\square$

We conclude this section with an extension of isotopies theorem.

5.6 Theorem.

Let M be a connected, second countable manifold, and let $\mu_o \in M_g^{\partial}(M)$ with $\mu_o(M) < \infty$.

Let $C \subset M$ be a compact subset such that $C(M \setminus C)$ is finite, and let $U \subset M$ be a neighbourhood of C. Let

$$\iota : I \to I(U, M; \mu_o)_K$$
$$\tau \mapsto \iota_\tau$$

be an isotopy of embeddings preserving μ_o such that ι_o is the inclusion $\eta : U \hookrightarrow M$.

Then the following conditions are equivalent:

(1) $\tau \mapsto \iota_\tau|_C$ extends to a compactly supported isotopy

$\tilde{\iota} : I \to H_c(M, \mu_o)$ of measure preserving homeomorphisms such that $\tilde{\iota}_o = Id_M$.

(2) If $\tilde{\iota} : I \to H_c(M, \mu_o\text{-e-reg})$ is any extension of $\tau \mapsto \iota_\tau|_C$ to biregular homeomorphism such that $\tilde{\iota}_o = Id_M$, then $\mu_o(\tilde{\iota}_\tau(A)) = \mu_o(A)$ for each $A \in C(M \setminus C)$.

Proof.

Observe that if condition (2) is satisfied by some extension of $\tau \mapsto \iota_\tau|_C$, then, by Remarks 5.3, it is satisfied by all. Therefore (1) implies (2).

Let $C^+ \subset \overset{\circ}{U}$ be a compact neighbourhood of C. Then, from Theorem 4.7, it follows that a compactly supported extension of $\tau \mapsto \iota_\tau|_{C^+}$ to biregular homeomorphisms does exist (see Edwards and Kirby [7 , p. 79, proof of corollary 1.2]).
Then we can modify this biregular extension to a measure preserving extension of $\tau \mapsto \iota_\tau|_C$ as in Proposition 5.2 .

□

5.7 Remark.

Suppose that $\tau \mapsto \iota_\tau$ can be extended to a measure preserving isotopy of homeomorphisms. Furthermore, suppose that $D \subset \partial M$ closed and a neighbourhood V of D in ∂M are given in such a way that ι_τ fixes $U \cap V$ pointwise, then $\tilde{\iota}_\tau$ can be chosen to fix D pointwise for each $\tau \in I$.

Chapter II. *The algebra.*

§6. A homology theory based on measures, due to W.Thurston.

In this section we discuss the domain and range of the mass flow homomorphism.

6.1 The Universal Covering of Some Groups of Homeomorphisms

A *topological pair* (X, A) consists of a topological space X and a subspace $A \subset X$. If A is empty we shall not distinguish between the pair (X, ϕ) and the space X. A map $f : (X, A) \to (Y, B)$ between pairs is a continuous function f from X into Y such that $f(A) \subset B$.

Given a pair (X, A), we let $(X, A) \times I$ denote the pair $(X \times I, A \times I)$. Let $X' \subset X$ and suppose that $f_0, f_1 : (X, A) \to (Y, B)$ agree on X'. Then f_0 is *homotopic to* f_1 *relative to* X' if there exists a map

$$F : (X, A) \times I \to (Y, B)$$

such that

$$F(x, o) = f_0(x) \qquad \text{for } x \in X ;$$

$$F(x, 1) = f_1(x) \qquad \text{for } x \in X ;$$

$$F(x, t) = f_0(x) \qquad \text{for } (x, t) \in X' \times I.$$

Such a map F is called a *homotopy relative to* X' *from* f_0 *to* f_1

Let x_0 be a point in a space X. Denote by $P(X, x_0)_K$ the space of all *paths* γ *based at* x_0 (i.e. the space of continuous functions γ from $(I, \{o\})$ into $(X, \{x_0\})$) endowed with the compact-open topology, and let $\Omega(X, x_0)_K$ denote the subspace of *loops* (i.e. the space of continuous functions from $(I, \partial I)$ into $(X, \{x_0\})$).

Let $\widetilde{(X, x_0)}$ denote the space of equivalence classes of paths in $P(X, x_0)$ (under the relation of homotopy relative to ∂I) endowed with the quotient topology.

The continuous map $ev : P(X, x_0)_\kappa \to X$ defined by $ev(\gamma) = \gamma(1)$ for each $\gamma \in P(X, x_0)$ induces a (continuous) function $\rho : \widetilde{(X, x_0)} \to X$ such that $\rho([\gamma]) = \gamma(1)$, where $[\gamma] \in \widetilde{(X, x_0)}$ denotes the equivalence class of $\gamma \in P(X, x_0)$.

Observe that $\rho^{-1}(\{x_0\})$ is the (underlying set of the) fundamental group $\pi(X, x_0)$ of X at x_0.

If X is a connected, locally connected, locally semi-simply connected space, then $\widetilde{(X, x_0)}$ is (a model of) the universal covering space of X(see Spanier [23 , Chapter 2]).

If there is no possibility of confusion, we shall write $P(X)$, $\Omega(X)$, $\tilde{X}$ and $\pi(X)$ for $P(X, x_0)$, $\Omega(X, x_0)$, $\widetilde{(X, x_0)}$ and $\pi(X, x_0)$ respectively.

If $f : (X, \{x_0\}) \to (Y, \{y_0\})$ is a continuous mapping of (pointed) spaces, then we have the following commutative diagram of continuous functions,

$$\begin{array}{ccc}
P(X, x_0)_\kappa & \xrightarrow{P(f)} & P(Y, y_0)_\kappa \\
\downarrow & & \downarrow \\
\widetilde{(X, x_0)} & \xrightarrow{\tilde{f}} & \widetilde{(Y, y_0)} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}$$

where $P(f)(\gamma) = f \circ \gamma$ and $\tilde{f}([\gamma]) = [f \circ \gamma]$ for each $\gamma \in P(X, x_0)$.

Before concluding this paragraph we want to state a few facts about the behavior of topological groups and some stratified spaces under the above (functorial) constructions.

We start by saying that $X \mapsto \tilde{X}$ is a functor on the category of topological groups:

(A) Let X be a topological group with identity e.. Then $P(X)_K = P(X, e)_K$ is a topological group having as identity the constant path $\eta : t \to e$ $(o \leq t \leq 1)$, and group operations given by the rules

$$P(X) \times P(X) \longrightarrow P(X)$$

$$(\sigma, \gamma) \longmapsto (t \mapsto \sigma(t)\,\gamma(t))\ ;$$

$$P(X) \longrightarrow P(X)$$

$$\sigma \longmapsto (t \mapsto (\sigma(t))^{-1})\ .$$

There is a unique group structure on $\tilde{X}$ making the canonical projection $P(X) \to \tilde{X}$ a group homomorphism. Also, with such a structure on $\tilde{X}$, the projection $\rho : \tilde{X} \to X$ becomes a homomorphism of topological groups from $\tilde{X}$ onto the path component of the identity in X.

The fundamental group $\pi(X)$ of X based at the identity is the kernel of the homomorphism $\rho : \tilde{X} \to X$.

If $f : X \to Y$ is a homomorphism of topological groups, then $P(f) : P(X) \to P(Y)$ and $\tilde{f} : \tilde{X} \to \tilde{Y}$ are also morphisms of topological groups.

Throughout the rest of this thesis we are mainly concerned with the group $H_{c,o}(M, \mu_o)$ of μ_o-measure preserving homeomorphisms of a manifold M which are compactly isotopic to the identity.

We have already seen that this group carries two topologies. Namely, the Whitney topology $\mathcal{W}$, and the $\varinjlim$-topology.

The following discussion tells us that the relationship between these topologies is preserved at "the universal covering level".

(B) Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a coherent family of metric spaces on X such that

(1) $x_0 \in A_\alpha$ for each $\alpha \in \Lambda$;

(2) $\{A_\alpha\}_{\alpha \in \Lambda}$ has a countable cofinal subfamily ;

(3) X is Hausdorff in the $\varinjlim$-topology. Then $\{ P(A_\alpha)_\kappa \mid \alpha \in \Lambda \}$ is a coherent family of metric (hence compactly generated) spaces on $P(X_{\varinjlim})$ and the identity function

$P(X_{\varinjlim})_{\varinjlim} \xhookrightarrow{\text{Id}} P(X_{\varinjlim})_\kappa$ is a continuous proper bijection of Hausdorff spaces (it is proper because $P(X_{\varinjlim})_\kappa \times I \to X_{\varinjlim}$, $(\gamma, t) \mapsto \gamma(t)$ is continuous), hence $k(P(X_{\varinjlim})_\kappa) = P(X_{\varinjlim})_{\varinjlim}$ (see Appendix One (4), (5) and Appendix Two (10), (11)).

Furthermore, if there is a Hausdorff topology τ on X compatible with each A_α (i.e. making each inclusion $A_\alpha \hookrightarrow X_\tau$ a closed embedding) such that each compact subset K of X is contained in some A_α (i.e. $X_{\varinjlim} \xhookrightarrow{\mathrm{Id}} X_\tau$ is a proper map), then the inclusion $P(X_{\varinjlim})_\kappa \to P(X_\tau)_\kappa$ is again a continuous proper bijection, so $k(P(X_{\varinjlim})_\kappa) = k(P(X_\tau)_\kappa)$.

Suppose now that the space $kX_\tau = X_{\varinjlim}$ is connected and locally contractible in such a way that for each $x \in X$ there is a neighbourhood U of x in X_τ such that $U_{\varinjlim}$ (i.e. U endowed with the subspace topology from $X_{\varinjlim}$) contracts into $\{x\}$ by a contraction fixing x. Then X_τ is locally semi-simply connected and we have the following (continuous) commutative diagram,

$$
\begin{tikzcd}
k(P(X_\tau)_\kappa) \arrow[r, hook, "\text{proper map}"'] \arrow[d] & P(X_{\varinjlim})_\kappa \arrow[r, hook, two heads, "\mathrm{Id}", "\text{proper map}"'] \arrow[d, "\text{open map}"] & P(X_\tau)_\kappa \arrow[d, "\text{open map}"] \arrow[dd, bend left=60, "\text{fibration}"] \\
k(\tilde{X}_\tau) \arrow[r, equal] \arrow[d] & \tilde{X}_{\varinjlim} \arrow[r, hook, "\mathrm{Id}", "\text{proper map}"'] \arrow[d, "\text{universal covering}"] & \tilde{X}_\tau \arrow[d, "\text{universal covering}"] \\
k(X_\tau) \arrow[r, equal] & X_{\varinjlim} \arrow[r, hook, two heads, "\mathrm{Id}", "\text{proper map}"'] & X_\tau
\end{tikzcd}
$$

As an example of this situation we can let X_τ to be $H_{c,o}(M, \mu_o)_m$, where M is a second countable manifold and μ_o is a measure in $M_g^\partial(M)$. Also $H_{c,o}(M)_m$, $H^\partial_{c,o}(M)_m$ and $H^\partial_{c,o}(M, \mu_o)_m$ are suitable examples.

Finally, we want to establish the difference between the concepts of "isotopies" and "paths of homeomorphisms".

(c) Let Y be a locally compact, locally connected, Hausdorff space. Define

$$\begin{aligned} IS(Y) = \{h : Y \times I \to Y \mid\ & h \text{ is continuous and, for each } t \in I, \\ & \text{the map } h_t : Y \to Y,\ y \mapsto h(y, t) \text{ is} \\ & \text{a homeomorphism and } h_0 = \mathrm{Id}\ \}. \end{aligned}$$

If $IS(Y)$ is given the compact open topology, then by the "exponential law" the resulting space is canonically homeomorphic to the space of paths of homeomorphisms $P(H(Y)_K)_K$, and either may be called "the space of isotopies of Y" (see 1.13).

In the case of homeomorphisms with compact support and compactly supported isotopies, the situation is slightly different: Define

$$\begin{aligned} IS(K, Y) &= \{h \in IS(Y) \mid \operatorname{supp} h_t \subset K \text{ for each } t \in I\ \} \text{ for } \\ & \qquad K \subset Y \text{ compact ;} \end{aligned}$$

$$IS_c(Y) = \cup \{IS(K, Y) \mid K \subset Y \text{ is compact }\}.$$

Then the following properties hold,

(1) $IS(K, Y)$ is closed in $IS_c(Y)_K$;

(2) There is a canonical continuous bijection from $P(H_c(Y)_{\lim\limits_{\to}})_K$ onto $IS_c(Y)_K$ which maps the closed subspace

$P(H(K, Y)_K)_K$ homeomorphically onto $IS(K, Y)_K$;

(3) $k(P(H_c(Y)_{\underset{\rightarrow}{\lim}})_K) \cong IS_c(Y)_{\underset{\rightarrow}{\lim}}$.

A similar analysis may be done for homeomorphisms preserving a good measure and homeomorphisms fixing the boundary of a manifold.

6.2 A Homology Theory Due To W. Thurston.

Let $\mathbb{R}^\infty$ be the vector space of all sequences $\{x_i \mid i = 1,2,\ldots\} = x$ of real numbers which vanish from some point on; that is, there exists a non-negative integer n (depending on x) such that $x_i = o$ for all $i \geq n$.

Let e_i $(i = 1,2,\ldots)$ be the vector whose i-th coordinate is one and all other coordinates are zero, and let e_o be the zero vector.

Identify $\mathbb{R}^n (n \geq o)$ with the subspace of $\mathbb{R}^\infty$ having all components after the n-th equal to zero.

For $q \in \mathbb{N}$, define the standard (geometric) simplex Δ^q as the convex hull of the set $\{e_o, e_1, \ldots, e_q\}$ (note that Δ^1 is just the unit interval I).

If $P_o, P_1, \ldots, P_q$ are points in some vector space $\mathbb{E}$, $(P_o P_1 \ldots P_q)$ will denote the restriction to Δ^q of the unique affine map $\mathbb{R}^q \to \mathbb{E}$ taking e_i into P_i $(i = o,1,\ldots,q)$.

Given a space X, a singular q-simplex in X is a (continuous) map $\sigma : \Delta^q \to X$. Let $S(\Delta^q, X)_K$ denote the space of all singular q-simplices with the compact-open topology. Note that $S(\Delta^q, X)_K$ is second countable if X is second countable (see Dugundji [6, Chapter XII]).

Let $C_q X$ denote the real vector space of finite signed Borel measures on $S(\Delta^q, X)_K$ with compact support (i.e. $\mu \in C_q X$ if and only if μ is a σ-additive, real valued function defined on all Borel subsets of $S(\Delta^q, X)_K$ such that there exists a $K \subset S(\Delta^q, X)_K$ compact with the property that the complement of K in $S(\Delta^q, X)$ has zero μ measure).

The linear space $C_q X$ becomes a locally convex topological vector space if it is given the weakest topology such that, for each $\lambda : S(\Delta^q, X)_K \to \mathbb{R}$ continuous, the functional

$$C_q X \to \mathbb{R}$$
$$\nu \mapsto \int \lambda \, d\nu$$

is continuous.

For $q > o$ and $o \le i \le q$, define the i-th <u>face of</u> Δ^q as the affine map $F_q^i : \Delta^{q-1} \hookrightarrow \Delta^q$ such that

$$F_q^i(e_j) = \begin{cases} e_j & j < i \\ e_{j+1} & j \ge i \end{cases},$$

(F_q^i can also be denoted by $(e_0, \dots, \hat{e}_i, \dots, e_q)$).

There is a continuous i-th <u>face map on singular</u> q-<u>simplices</u> defined by

$$(F_q^i)^* : S(\Delta^q, X) \to S(\Delta^{q-1}, X)$$

$$\sigma \mapsto \sigma \circ F_q^i$$

which induces a linear (continuous) transformation

$$\partial_q^i : C_q X \to C_{q-1} X$$

such that $\partial_q^i(\nu) = [(F_q^i)^*]_* \nu$ for each $\nu \in C_q X$, where $\partial_q^i(\nu)(B) = \nu([(F_q^i)^*]^{-1}(B))$ for each Borel subset B contained in $S(\Delta^{q-1}, X)_K$:

Since $\partial_{q-1}^j \circ \partial_q^i = \partial_{q-1}^{i-1} \circ \partial_q^j$ for $j < i$, the <u>boundary operator</u>

$$\partial_q = \sum_{i=0}^{q} (-1)^i \partial_q^i$$

is such that $\partial_{q-1} \circ \partial_q = 0$.

Now let $f : X \to Y$ be a continuous map. Then, for $q \in \mathbb{N}$, the function

$$f_q : S(\Delta^q, X)_K \to S(\Delta^q, Y)_K$$

$$\sigma \mapsto f \circ \sigma$$

is continuous. Hence the rule

$$f_{q*} : C_q X \to C_q Y$$

$$\nu \mapsto f_{q*}(\nu)$$

defines a continuous linear transformation (note that if $K \subset S(\Delta^q, X)$ supports ν, then $f_q(K)$ supports $f_{q*}\nu$).

Furthermore, the following relations are satisfied,

(1) $f_{(q-1)*} \circ \partial_q = \partial_q \circ f_{q*}$

(2) $g_{q*} \circ f_{q*} = (g \circ f)_{q*}$

Summarizing, we have defined a covariant functor C from the category of topological spaces to the category of (non-negative) chain complexes over $\mathbb{R}$, which assigns to each space X the complex $(CX, \partial) = \{(C_q X, \partial_q) \mid q \in \mathbb{N}\}$.

The composite of the functor C and the homology functor H is a covariant functor from the category of spaces to the category of graded vector spaces over $\mathbb{R}$ and homomorphisms of degree zero (see Spanier [23 , Chapter 4]).

If A is a subspace of X, there is a relative chain complex $(C(X, A), \bar{\partial}) = \{(C_q(X, A), \bar{\partial}_q) \mid q \in \mathbb{N}\} = \{(C_qX/C_qA, \bar{\partial}_q) \mid q \in \mathbb{N}\}$,

where, for each $q \in \mathbb{N}$, $\bar{\partial}_q$ is the unique map which makes the following diagram commutative,

$$\begin{array}{ccccc}
C_q A & \hookrightarrow & C_q X & \twoheadrightarrow & C_q X/C_q A \\
\downarrow \partial_q & & \downarrow \partial_q & & \downarrow \bar{\partial}_q \\
C_{q-1} A & \hookrightarrow & C_{q-1} X & \twoheadrightarrow & C_{q-1} X/C_{q-1} A
\end{array} .$$

Consequently, there is a relative homology (graded) vector space $H(C(X, A), \bar{\partial})$ of X modulo A. The relative homology vector space is a covariant functor from the category of topological pairs to the category of graded vector spaces over $\mathbb{R}$.

If X is a topological space, there is a natural identification of the space of zero-singular simplices $S(\Delta^o, X)_\kappa$ with X by associating to each simplex $\sigma : \Delta^o \to X$ the point $\sigma(o)$ in X. In this manner C_oX becomes the linear space of all signed Borel measures on X with compact support and

$$\varepsilon : C_o X \to \mathbb{R}$$

$$\nu \mapsto \nu(X)$$

is an augmentation (i.e. an epimorphism such that the composite $C_1 X \xrightarrow{\partial_1} C_o X \xrightarrow{\varepsilon} \mathbb{R}$ is trivial). Indeed, for if $\nu \in C_1 X$, then $\varepsilon(\partial_1\nu) = \varepsilon(\partial_1^0 (\nu)) - \varepsilon(\partial_1^1(\nu)) = \nu(S(\Delta^1, X)) - \nu(S(\Delta^1, X)) = o$.

Let X be a topological space, and let $(\Delta(X, \mathbb{R}), \partial) = \{(\Delta_q(X, \mathbb{R}), \partial_q) \mid q \in \mathbb{N} \}$ be the standard singular real chain complex associated with X (in particular $\Delta_q(X, \mathbb{R})$ is the vector space of formal finite linear combinations of singular q-simplices with real coefficients).

The space $\Delta_q(X, \mathbb{R})$ can be naturally embedded in $C_q X$ by sending each simplex $\sigma : \Delta^q \to X$ to the atomic probability supported in $\{\sigma\}$ and then extending linearly to all of $\Delta_q(X, \mathbb{R})$.

The inclusions

$$i_X : \Delta(X, \mathbb{R}) \hookrightarrow C X \quad \text{for all topological spaces } X,$$

certainly define a natural chain map preserving augmentation.

It can be proved that the homology functor $H \circ C$ that we have just constructed defines a homology theory with compact supports in the sense of Eilenberg and Steenrod for the category of normal topological Hausdorff pairs (X, A), and that the above inclusion i_X induces an isomorphism

$$H(i_X) : H(\Delta(X, \mathbb{R}), \partial) \cong H(CX, \partial) \quad \text{for every polyhedron } X.$$

This result is extended to absolute neighbourhood retracts (ANR's) and in particular to manifolds, since an ANR has the homotopy type of a polyhedron (see Lundell and Weingram [17 , Theorem 3.8, p. 127]).

Nevertheless, we will distinguish these homology theories and denote by $H_q^s(X, \mathbb{R})$ the q-th singular homology group with real coefficients of the space X, and by $H_q(X, \mathbb{R})$ the q-th homology $\mathbb{R}$-vector space of X defined above.

6.3 Examples.

The first of our examples is just the unit circle. The computations performed on the "infinite telescope" described afterwards will allow us, in 6.5, to show that the first homology space is separated.

I want to thank D.B.A. Epstein for pointing out to me the relevance of this beautiful space to the problem.

(1) An Eilenberg-Mac Lane space of type $(1, \mathbb{Z})$.

Let T^1 denote the topological group $\mathbb{R}/\mathbb{Z}$, which is isomorphic to the unit circle $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$. We want to construct an explicit isomorphism of the first homology vector space $H_1(T^1, \mathbb{R})$ onto $\mathbb{R}$.

For this purpose let

$$\lambda : S(\Delta^1, T^1)_\kappa \to \mathbb{R}$$

be the continuous map defined by

$$\lambda(\sigma) = \bar{\sigma}(1) - \bar{\sigma}(0) \qquad \text{for all } \sigma : \Delta^1 \to T^1 \quad \text{continuous,}$$

where $\bar{\sigma} : \Delta^1 \to \mathbb{R}$ is any lifting of σ. Alternatively, $\lambda(\sigma) = [\overline{\sigma - \sigma(0)}\,](1)$, where $\overline{\sigma - \sigma(0)} : \Delta^1 \to \mathbb{R}$ is the lifting of $\sigma - \sigma(0)$ such that $[\overline{\sigma - \sigma(0)}\,](0) = 0$.

Now we define two linear functionals

$$D : C_1 T^1 \to \mathbb{R}$$

such that

$$D(\nu) = \int \lambda \, d\nu \qquad \text{for all } \nu \in C_1 T^1$$

and

$$d : \Delta_1 (T^1, \mathbb{R}) \to \mathbb{R}$$

determined by the formula

$$d(\sigma) = \lambda(\sigma) \qquad \text{for all } \sigma \in S(\Delta^1, T^1).$$

(recall that $\Delta_1(T^1, \mathbb{R})$ is the free $\mathbb{R}$-linear space over $S(\Delta^1, T^1)$).

(*) | Note that if σ is a closed curve in T^1, then $d(\sigma)$ is just the degree of σ.

(A) Assertion. Both d and D are zero on boundaries.

Proof.

Let $\sigma \in S(\Delta^2, T^1)$ and let $\bar{\sigma} : \Delta^2 \to \mathbb{R}$ be any lifting of σ. Then

$$\partial_2\sigma = \exp(\bar{\sigma} \circ F_2^0 - \bar{\sigma} \circ F_2^1 + \bar{\sigma} \circ F_2^2)$$

and

$$\begin{aligned} d\partial_2\sigma &= \bar{\sigma} \circ F_2^0(1) - \bar{\sigma} \circ F_2^0(0) - \bar{\sigma} \circ F_2^1(1) + \bar{\sigma} \circ F_2^1(0) + \\ &+ \bar{\sigma} \circ F_2^2(1) - \bar{\sigma} \circ F_2^2(0) = \bar{\sigma}(e_2) - \bar{\sigma}(e_1) - \bar{\sigma}(e_2) + \\ &+ \bar{\sigma}(e_0) + \bar{\sigma}(e_1) - \bar{\sigma}(e_0) = 0 \\ &= (\lambda \circ (F_2^0)^* - \lambda \circ (F_2^1)^* + \lambda \circ (F_2^2)^*)\,\sigma \end{aligned}$$

Now let $\nu \in C_2T^1$; then

$$D\,\partial_2\nu = \int_{S(\Delta^1, X)} \lambda\, d\,\partial_2\,\nu$$

$$= \int_{S(\Delta^2, X)} (\lambda\circ(F_2^0)^* - \lambda\circ(F_2^1)^* + \lambda\circ(F_2^2)^*)d\nu$$

but $\lambda\circ(F_2^0)^* - \lambda\circ(F_2^1)^* + \lambda\circ(F_2^2)^* \equiv 0$ as the previous computation has shown.

(B) Assertion. Let $i : \Delta_1(T^1, \mathbb{R}) \to C_1T^1$ be the canonical injection. Then

$$D \circ i = d .$$

Proof.

Let $\sigma : \Delta^1 \to T^1$ be a generator in $\Delta_1(T^1, \mathbb{R})$. Then

$$D \circ i(\sigma) = \int \lambda\, d\, i(\sigma) = \lambda(\sigma) = d(\sigma)$$

(C) By assertions (A) and (B), the maps d, D and i induce the following commutative diagram in homology

$$\begin{tikzcd} H_1^s(T^1, \mathbb{R}) \arrow[rr, "i_*"] \arrow[dr, "\bar{d}"'] & & H_1(T^1, \mathbb{R}) \arrow[dl, "\bar{D}"] \\ & \mathbb{R} & \end{tikzcd}$$

But i_* is an isomorphism and $\bar{d}$ is also an isomorphism by (*) above. Hence $\bar{D}$ is an isomorphism as well.

(2) An Eilenberg - Mac Lane space of type (1, Q)

> I have a telescope which I keep in a cylindrical box. The box contains an infinite number of segments. When you pull them out, they click along its edges in a rather peculiar way.
> You must look through the telescope from the wrong end. But then, you can see the rationals.

Let j be a positive integer and let T^1_j denote the topological group $\mathbb{R}/(1/j!\ \mathbb{Z})$, where $j! = 1\cdot 2\cdot 3\cdot \ldots \cdot j$ and $1/j!\,\mathbb{Z}$ denotes the discrete group of all integral multiples of $1/j!$.

If $o < j \leq \ell$ then there is a natural group homomorphism

$$T^1_j \to T^1_\ell$$

$$x + 1/j!\,\mathbb{Z} \mapsto x + 1/\ell!\,\mathbb{Z}$$

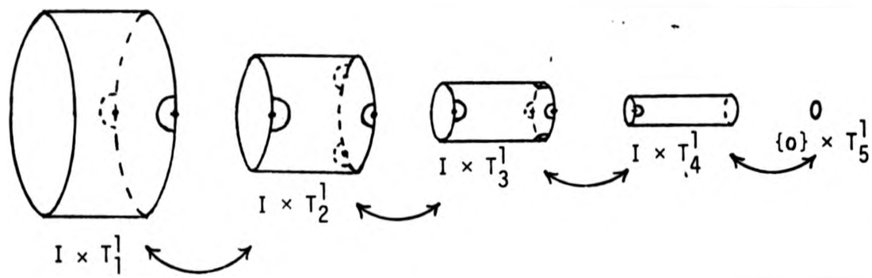
Observe that, in particular, the kernel of $T^1_j \to T^1_{j+1}$ is cyclic of order $j + 1$.

Let ℓ be a positive integer and define $K(1, Q)_\ell$ as the quotient space obtained from the disjoint union

$$I \times T^1_1 \cup I \times T^1_2 \cup \ldots \cup I \times T^1_\ell \cup \{o\} \times T^1_{\ell+1}$$

by identifying the point $(1, x)_j = (1, x + 1/j!\,\mathbb{Z}) \in I \times T^1_j$ with $(o, x)_{j+1} = (o, x + 1/(j+1)!\,\mathbb{Z}) \in I \times T^1_{j+1}$ (i.e. $(1, x)_j \sim (o, x)_{j+1}$).

If $(t, x)_j \in I \times T^1_j$ then $[t, x]_j$ will denote the corresponding point in $K(1, Q)_\ell$.



$K(1, Q)_4$

Observe that $K(1, Q)_\ell$ has a "horizontal" axis and that we can deform the space in itself by "pushing to the right" along this axis. Therefore, $\{o\} \times T^1_{\ell+1}$ is a deformation retract of $K(1, Q)_\ell$.

An explicit retraction by deformation for the Mobius strip $K(1,Q)_1$ may be given by the formulas $([t, x]_1, s) \mapsto [t(1-s) + s, x]_1$ and $([o,x]_2, s) \mapsto [o, x]_2$.

In particular, the function

$$\zeta_\ell : K(1, Q)_\ell \to T^1_{\ell+1}$$

$$[t, x]_j \mapsto x + 1/(\ell+1)!\,\mathbb{Z}$$

is a homotopy equivalence, showing that the fundamental group of $K(1, Q)_\ell$ is infinite cyclic and that the circle $\{o\} \times T^1_{\ell+1}$ (when traversed in the positive direction) defines a generator of the group.

There is a natural embedding

$$K(1, Q)_\ell \hookrightarrow K(1, Q)_{\ell+1}$$

$$[t, x]_j \mapsto [t, x]_j$$

Finally, we define the space

$$K(1, Q) = \{\coprod_{j=1}^{\infty} I \times T^1_j\} \;/\; ((1, x)_j \sim (o, x)_{j+1})$$

and note that the topology of $K(1, Q)$ as a quotient space coincides with the topology coinduced by the family $\{ K(1, Q)_\ell \mid \ell = 1, 2, \ldots \}$.

The commutative diagram

$$\begin{tikzcd}
T^1_2 \arrow[r] & T^1_3 \arrow[r] & \cdots \arrow[r] & T^1_{\ell+1} \arrow[r] & \cdots \\
K(1, Q)_1 \arrow[u, "\zeta_1"'] \arrow[r, hook] & K(1, Q)_2 \arrow[u, "\zeta_2"'] \arrow[r, hook] & \cdots \arrow[r, hook] & K(1, Q)_\ell \arrow[u, "\zeta_\ell"'] \arrow[r, hook] & \cdots
\end{tikzcd}$$

induces

a commutative diagram

$$\begin{array}{ccccccc}
1/2!\,\mathbb{Z} & \hookrightarrow & 1/3!\,\mathbb{Z} & \hookrightarrow \dots \hookrightarrow & 1/(\ell+1)!\,\mathbb{Z} & \hookrightarrow & \dots \\
\wr\| & & \wr\| & & \wr\| & & \\
\pi(K(1,Q)_1) & \hookrightarrow & \pi(K(1,Q)_2) & \hookrightarrow \dots \hookrightarrow & \pi(K(1,Q)_\ell) & \hookrightarrow & \dots
\end{array}$$

It easily follows that there is a group isomorphism from $\pi(K(1, Q))$ onto the group Q of rational numbers.

The fact that the higher homotopy groups of a circle vanish, imply that the space we have defined has trivial higher homotopy groups (for any continuous $\alpha: S^n \to K(1, Q)$ has its image in one $K(1, Q)_\ell$).

Now we want to exhibit an isomorphism of the homology vector space $H_1(K(1, Q), \mathbb{R})$ onto $\mathbb{R}$.

For this purpose we observe first that the inclusion $K(1, Q)_\ell \hookrightarrow K(1, Q)_{\ell+1}$ can be lifted to a map from the universal covering $\tilde{K}(1, Q)_\ell$ of $K(1, Q)_\ell$ into the universal covering $\tilde{K}(1, Q)_{\ell+1}$ of $K(1, Q)_{\ell+1}$, and that the continuous function $\zeta_\ell : \tilde{K}(1, Q)_\ell \to T^1_{\ell+1}$ can be lifted to a map $\tilde{\zeta}_\ell: \tilde{K}(1, Q)_\ell \to \mathbb{R}$ in such a way that the following diagram commutes

$$
\begin{tikzcd}
 & \tilde{K}(1, Q)_{\ell+1} \arrow[rr, "\tilde{\zeta}_{\ell+1}"] \arrow[dd] & & \mathbb{R} \arrow[dd] \\
\tilde{K}(1, Q)_{\ell} \arrow[ur] \arrow[rr, "\tilde{\zeta}_{\ell}"] \arrow[dd] & & \mathbb{R} \arrow[ur, "\text{Identity}"] \arrow[dd] & \\
 & K(1, Q)_{\ell+1} \arrow[rr, "\zeta_{\ell+1}"] & & T^{1}_{\ell+2} \\
K(1,Q)_{\ell} \arrow[ur] \arrow[rr, "\zeta_{\ell}"] & & T^{1}_{\ell+1} \arrow[ur] &
\end{tikzcd}
$$

Therefore, there is a well defined continuous map

$$\lambda : S(\Delta^{1}, K(1, Q))_{\kappa} \rightarrow \mathbb{R}$$

defined by the formula

$$\lambda(\sigma) = \tilde{\zeta}_{\ell} \circ \bar{\sigma}(1) - \tilde{\zeta}_{\ell} \circ \bar{\sigma}(0) \ ,$$

where ℓ is any integer such that $\sigma(\Delta^{1})$ is contained in $K(1, Q)_{\ell}$ and $\bar{\sigma}: \Delta^{1} \rightarrow \tilde{K}(1, Q)_{\ell}$ is any lifting of σ.

We define two linear functionals,

$$D : C_{1} K(1, Q) \rightarrow \mathbb{R}$$

defined by the integral expression

$$D(\nu) = \int \lambda d\nu \qquad \text{for all } \nu \in C_{1}K(1, Q)$$

and

$$d : \Delta_{1}(K(1, Q), \mathbb{R}) \rightarrow \mathbb{R}$$

determined by the formula

$$d(\sigma) = \lambda(\sigma) \qquad \text{for all } \sigma \in S(\Delta^1, K(1, Q)).$$

Using the previous example it is not difficult to see that both D and d are zero on boundaries and that the following induced diagram in homology commutes,

$$\begin{tikzcd} H_1^S(K(1, Q), \mathbb{R}) \arrow[rr, "i_*"] \arrow[dr, "\bar{d}"'] & & H_1(K(1, Q), \mathbb{R}) \arrow[dl, "\bar{D}"] \\ & \mathbb{R} & \end{tikzcd}$$

In the same way as in example (1), it follows that $\bar{D}$ is an isomorphism.

6.4 <u>Discussion.</u>

Let X be a metric space and let q be a non-negative integer. Recall that the space C_qX of q-chains has been given a locally convex Hausdorff topology, namely, the weak topology which, for each continuous $\lambda : S(\Delta^q, X)_\kappa \to \mathbb{R}$, makes the functional

$$C_q X \to \mathbb{R}$$

$$\nu \mapsto \int \lambda \, d\nu$$

continuous.

It is easily verified that $C_q X$ is Hausdorff.

Let $Z_q X$ and $B_q X$ denote the topological vector subspace of q-cycles and the subspace of q-boundaries respectively.

Endow $H_q(X, \mathbb{R}) = Z_q X/B_q X$ with the quotient topology. As in any quotient topological vector spaces, $H_q(X, \mathbb{R})$ is Hausdorff if and only if $B_q X$ is closed in $Z_q X$. Observe that Z_q is the kernel of a boundary operator and therefore closed in $C_q X$.

> <u>Question</u>:
>
> Is $H_q(X, \mathbb{R})$ a
>
> Hausdorff space in general?

For example, if we let X be the space $K(1, Q)$ of the above paragraph, then the continuous commutative diagram (see 6.3(2))

$$\begin{tikzcd}
Z_1 X \arrow[d] \arrow[dr, "D"] & \\
H_1(X, \mathbb{R}) \arrow[r, "\bar{D}", "\cong"'] & \mathbb{R}
\end{tikzcd}$$

shows that the space $B_1 X$ is equal to the kernel of D. Hence, in this case, $H_1(X, \mathbb{R})$ is Hausdorff.

Furthermore, using the fact that any linear bijection from a finite dimensional, Hausdorff topological vector space over $\mathbb{R}$ onto some euclidean space is also a topological isomorphism, we can conclude that $\bar{D}$ is a homeomorphism as well.

This particular example suggests that the answer to the above question is affirmative for the first homology vector space. This is the content of the next Theorem.

6.5 Theorem

Let X be a metric absolute neighbourhood retract. Then $H_1(X, \mathbb{R})$ is a Hausdorff space.

Proof.

It is enough to prove that for each $z \in H_1(X, \mathbb{R})$ with z different than zero, there exists a continuous linear functional $\Lambda : H_1(X, \mathbb{R}) \to \mathbb{R}$ such that $\Lambda(z) \neq o$.

Let $[X, K(1, Q)]$ denote the (abelian group) of homotopy classes of continuous functions from X into $K(1, Q)$ and let $H_1^S(X, Q)$ be the first singular homology group of X with rational coefficients.

Since X is an ANR, obstruction theory (see Spanier [23, Chapter 8, section 1]) says that there is a one to one natural correspondence.

(1) $[X, K(1, Q)] \hookrightarrow\!\!\!\!\twoheadrightarrow \mathrm{Hom}(H_1^S(X, Q), Q)$ from $[X,K(1,Q)]$ onto the dual space $\mathrm{Hom}(H_1^S(X, Q), Q)$ of Q-linear homomorphisms of $H_1^S(X, Q)$ into Q, which assigns to each class $[f]$ of a continuous map $f : X \to K(1, Q)$ the homomorphism $H_1(f) = H_1^S(f, Q): H_1^S(X, Q) \to Q = H_1^S(K(1, Q), Q)$.

Put $V = H_1^S(X, Q)$. Then V is naturally embedded in its double dual by the funtion

$$(2) \quad V \xhookrightarrow{i} \mathrm{Hom}(\mathrm{Hom}(V,\ Q),\ Q)$$

$$z \longmapsto (\zeta \mapsto \zeta(z))\ .$$

Tensoring by $\mathbb{R}$ the above homomorphism gives an embedding

$$(3) \quad V \otimes \mathbb{R} \xhookrightarrow{i \otimes \mathrm{Id}} \mathrm{Hom}(\mathrm{Hom}(V,\ Q), Q) \otimes \mathbb{R}\ .$$

By composing $i \otimes \mathrm{Id}$ with the inclusion

$$(4) \quad \mathrm{Hom}(\mathrm{Hom}(V,\ Q),\ Q) \otimes \mathbb{R} \hookrightarrow \mathrm{Hom}(\mathrm{Hom}(V,Q), \mathbb{R})$$

and using (1), we can construct a commutative diagram

$$\begin{array}{ccc}
H_1(X, \mathbb{R}) \cong H_1^s(X, \mathbb{R}) & \hookrightarrow & \mathrm{Hom}([X,\ K(1,Q)]\ , \mathbb{R}) \\
\wr\| & \text{Universal Coefficients} & \wr\| \\
H_1^s(X,\ Q) \otimes \mathbb{R} & \hookrightarrow & \mathrm{Hom}(\mathrm{Hom}(H_1^s(X,Q),\ Q), \mathbb{R})
\end{array}$$

which shows, in particular, that for every $z \in H_1(X, \mathbb{R})$ with $z \neq o$ there is a continuous $f : X \to K(1,\ Q)$ such that $H_1(f)z \neq o$. This concludes the proof.

$\square$

§7 The mass flow homomorphism.

In the paragraphs that follow we define the mass flow homomorphism $\tilde{\theta}$ and establish some of its properties. We first construct a continuous map $\bar{\Phi}$ from the space of paths $\mathcal{P}(\mathcal{H}_{c,o}(X, \mu_o)_m)$ into the vector space of 1-chains C_1X. Then we prove that $\bar{\Phi}(h)$ represents a well defined homology class $\tilde{\theta}([h])$ which does not depend on the homotopy class of the path h.

Afterwards, we state some basic facts about the mass flow. In particular, we compare our definition of $\tilde{\theta}$ with the mass flow homomorphism given by A. Fathi.

The rest of this section is devoted to study some aspects of a fundamental commutative diagram associated with the mass flow homomorphism.

7.1 Definition of The Mass Flow Homomorphism.

Throughout this paragraph X will denote a locally compact, locally connected, second countable Hausdorff space and μ_o will represent a fixed good measure on X.

(A) Let $\mathrm{Cont}(X, S(\Delta^1, X))$ be the set of all continuous functions from X into the space of 1-simplices $S(\Delta^1, X)$. Endow $\mathrm{Cont}(X, S(\Delta^1, X))$ with the Whitney topology m. Recall that a typical open basic set in this topology is given by the intersection $\bigcap_{i \in \Lambda} [K_i, U_i]$, where $\{K_i\}_{i \in \Lambda}$ is a locally finite family of compact sets in X, $\{U_i\}_{i \in \Lambda}$ is a family of open sets in $S(\Delta^1, X)$ and $[K_i, U_i]$ is the set of all continuous $f : X \to S(\Delta^1, X)$ with $f(K_i) \subset U_i$. Alternatively, if d is a metric on X a basic open neighbourhood of a function $f : X \to S(\Delta^1, X)$ may be given in terms of a continuous map $\delta : X \to \langle 0, \infty\rangle$ as the set $N_\delta(f)$ of all $g : X \to S(\Delta', X)$ such that $d(f_t(x), g_t(x)) < \delta(x)$ for each $x \in X$ and each $t \in I$.

From the second description of the Whitney topology just given, it follows that the (exponential) map

$$\begin{aligned} E : P(H(X)_m)_\kappa &\to \mathrm{Cont}(X, S(\Delta^1, X))_m \\ (1) \qquad h = \{h_t\}_{t \in I} &\mapsto (x \mapsto (t \mapsto h_t(x))) \end{aligned}$$

is a topological embedding.

Let H_τ be a group of homeomorphisms of X endowed with a topology τ such that the inclusion $H_\tau \overset{i}{\hookrightarrow} H(X)_m$ is continuous.

Then the composite

$$P(H_\tau) \xrightarrow{P(i)} P(H(X)_m)_\kappa \xrightarrow{E} \mathrm{Cont}(X, S(\Delta^1, X))_m$$

is continuous and will also be denoted by E.

Let $\eta \in P(H_\tau)_\kappa$ be the constant path such that

(2) $\eta_t = \mathrm{Id} : X \to X$ for each $t \in I$.

In what follows we shall reserve the letter E for the exponential map given in (1), and the letter η for the constant path in (2).

Observe that if $f: X \to S(\Delta^1, X)$ is continuous and a Borel measure ν on X is given, then $f_*\nu$ such that $f_* \nu(B) = \nu(f^{-1}(B))$, for each Borel subset B in $S(\Delta^1, X)$, is a Borel measure on $S(\Delta^1, X)$ with $\operatorname{supp} f_*\nu \subset f(\operatorname{supp} \nu)$.
Hence, if ν has compact support then $f_*\nu$ is a chain in $C_1 X$.

(B) <u>Assertion</u>: If $h \in P(H_c(X, \mu_0)_m)_\kappa$ and K is a compact subset of X containing the support supp h of the path h, then

$$E(h)_*(\mu_0|K) - E(\eta)_*(\mu_0|K)$$

$$= E(h)_*(\mu_0) - E(\eta)_*(\mu_0)$$

We give an informal proof:

The map $E(\eta) : X \to S(\Delta^1, X)$ is an embedding of X as a set of trivial simplices in $S(\Delta^1, X)$, which assigns to each $x \in X$ the constant singular 1-simplex $t \mapsto x$. Call the image of $E(\eta)$ the trivial copy of X in $S(\Delta^1, X)$.

The map $E(h) : X \to S(\Delta^1, X)$ is another embedding of X, whose image agrees with the trivial copy of X in $S(\Delta^1, X)$ except for a "bubble" lying over the trivial copy of supp h in $S(\Delta^1, X)$.

It follows that the measures $E(h)_*\mu_0$ and $E(\eta)_*\mu_0$ agree in the complement of the union of the trivial copy of supp h and the "bubble". This concludes the proof.

□

(C) By the above assertion we can define a function

$$\Phi = \Phi_{(X,\ \mu_0)} : P(H_{c,0}(X, \mu_0)_m)_K \to C_1X$$

such that

$$\Phi(h) = E(h)_*(\mu_0|_K) - E(\eta)_*(\mu_0|_K)$$

where K is any compact subset of X containing the support of the path h.

We want to prove now that Φ is continuous and that it induces a well defined map from $\tilde{H}_{c,0}(X, \mu_0)_m$ into $H_1(X, \mathbb{R})$.

This induced function is certainly the mass flow homomorphism.

Clearly, the term $E(\eta)_* \mu_o|_K$ must be homologically trivial, but it plays the technical role of making Φ a well defined function.

(D) Assertion. The map Φ is continuous.

Proof.

If $\zeta : S(\Delta^1, X) \to \mathbb{R}$ continuous is given, we have to check that the function

$$(1)\quad \begin{aligned} &P(H_{c,o}(X, \mu_o)_m)_K \to \mathbb{R} \\ &h \mapsto \int \zeta \, d\, \Phi(h) = \int [\, \zeta \circ E(h) - \zeta \circ E(\eta)\,]\, d\, \mu_o \end{aligned}$$

is continuous.

Recall that $E(h) : X \to S(\Delta^1, X)$ is an embedding of X in $S(\Delta^1, X)$. The map $S(\Delta^1, X) \to X$, such that $\sigma \mapsto \sigma(o)$, shows that X is actually embedded as a retract of $S(\Delta^1, X)$. Hence $E(h)$ is a closed embedding, so it is proper.

Let $\mathrm{Cont}_c(X, \mathbb{R})_m$ be the space of real valued functions on X with compact support endowed with the Whitney topology. Then the fact that $E(h)$ is proper for each path h, implies that the map

$$(2)\quad P(H_{c,o}(X, \mu_o)_m)_K \to \mathrm{Cont}_c(X, \mathbb{R})_m$$

given by the formula

$$h \mapsto \zeta \circ E(h) - \zeta \circ E(\eta)$$

is continuous.

Clearly, the function

(3) $\mathrm{Cont}_c(X, \mathbb{R})_m \to \mathbb{R}$

$f \mapsto \int_X f d\, \mu_o$

is continuous.

Since the composite of the maps given in (2) and (3) is the map given in (1), the assertion follows.

□

(E) Assertion. If $h \in P(H_{c,o}(X, \mu_o)_m)$, then $\Phi(h)$ is a one cycle, that is, $\partial_1(\Phi(h)) = o$.

Proof.

Let $K \subset X$ be a compact set such that $\mathrm{supp}\, h \subset K$. It is enough to prove that $\partial_1(E(h)_* \mu_o|_K) = o$.

By identifying the space of zero-singular simplices $S(\Delta^o, X)$ with X, the two 1-face maps on $S(\Delta^1, X)$ are

(1) $S(\Delta^1, X) \to X$

$\sigma \mapsto \sigma(1)$

and

(2) $S(\Delta^1, X) \to X$

$\sigma \mapsto \sigma(o)$

Now consider the composites of $E(h)$ with (1) and (2).

There are:

$$(3) \quad X \xrightarrow{E(h)} S(\Delta^1, X) \to X$$

$$x \mapsto h_1(x)$$

and

$$(4) \quad X \xrightarrow{E(h)} S(\Delta^1, X) \to X$$

$$x \mapsto x$$

Therefore, the 1-boundary of $E(h)_*(\mu_o|_K)$ must be the difference

$$h_{1*}\mu_o|_K - Id_*\,\mu_o|_K = \text{bdry}$$

But

$$\text{bdry} = \mu_o|_{h_1(K)} - \mu_o|_K$$

$$= \mu_o|_K - \mu_o|_K = o$$

because $K \supset \operatorname{supp} h_1$ and h_1 preserves μ_o.

□

(F) Remark. Loosely speaking, we can think that the image of $E(h) : X \hookrightarrow S(\Delta^1, X)$ is a " μ_o-measure preserving twisted cylinder" in $S(\Delta^1, X)$ having "X" as lower lid and "$h_1(X)$" as upper lid.

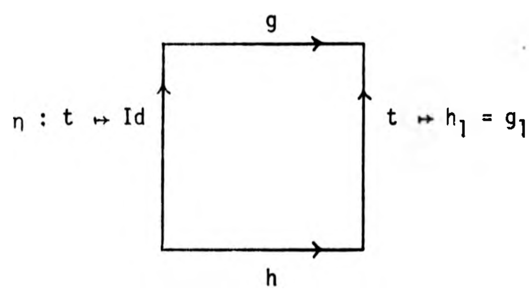
(G) Assertion. Let $h, g \in P(H_{c,o}(X, \mu_o)_m)$ be homotopic paths relative to ∂I.

Then $\Phi(h)$ is homologous to $\Phi(g)$ (i.e. $\Phi(h) - \Phi(g)$ is zero in $H_1(X, \mathbb{R})$).

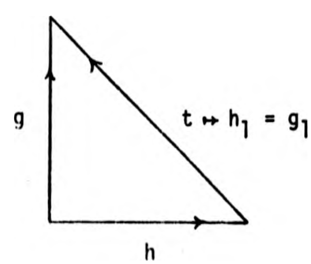
<u>Proof.</u>

Let $\mathcal{H}$ be $\mathcal{H}_{c,o}(X, \mu_o)_m$ and let $F : I \times I \to \mathcal{H}$ be a homotopy from h to g (rel ∂I).

Diagramatically, we have



Since F restricted to the edge $\{o\} \times I$ is constant, we can collapse $\{o\} \times I$ into a point and get a map from the standard 2-simplex Δ^2 into $\mathcal{H}$ given by the following diagram on the edges:



This map induces a continuous function from X into the space of

2-singular simplices on X, say

$$E_2(F) : X \to S(\Delta^2, X)$$

Let K be a compact subset of M containing $\operatorname{supp} F$. Now, computing

$$\partial_2(E_2(F)_* \, \mu_o|_K)$$

we see that it is equal to

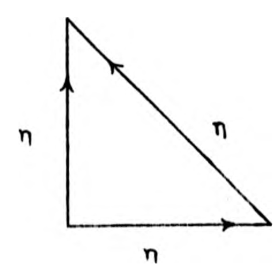
$$(E(\eta) \circ h_1)_* \, \mu_o|_K - E(g)_* \, \mu_o|_K + E(h)_* \, \mu_o|_K$$

and hence equal to

$$\Phi(h) - \Phi(g) + E(\eta)_*(h_{1*}\mu_o|_K)$$

$$= \Phi(h) - \Phi(g) + E(\eta)_* \, \mu_o|_K$$

By performing the same calculation using the constant map $\Delta^2 \to \{Id\} \subset H$, defined also by the diagram



we see that $E(\eta)_* \, \mu_o|_K$ is a boundary itself. This concludes the proof. □

Thus we are allowed to make the following important definition.

(H) Definition. Let the mass flow homomorphism

$$\tilde{\theta} = \tilde{\theta}_X = \tilde{\theta}_{(X,\mu_o)} : \tilde{H}_{c,o}(X,\mu_o)_m \to H_1(X, \mathbb{R})$$

be defined by the formula

$$\tilde{\theta}([h]) = [E(h)_* \, \mu_o|_{\text{supp } h}] ,$$

where $[h]$ is the homotopy class (rel ∂I) of the path $h \in P(H_{c,o}(X, \mu_o)_m)$ in $\tilde{H}_{c,o}(X,\mu_o)_m$ and $[E(h)_*\mu_o|_{\text{supp } h}]$ is the homology class of $\Phi(h)$ in $H_1(X,\mathbb{R})$.

From (D), it follows without effort that the mass flow homomorphism is a continuous function.

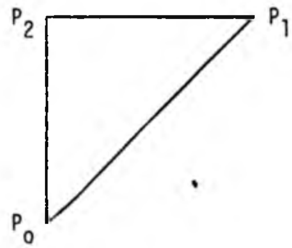
(I) Assertion. The mass flow

$$\tilde{\theta} : \tilde{H}_{c,o}(X, \mu_o)_m \to H_1(X, \mathbb{R})$$

is a group homomorphism.

Proof.

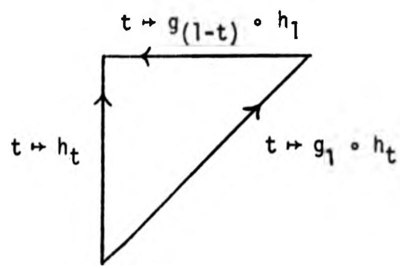
Let H be $H_{c,o}(X, \mu_o)_m$ and let Δ_1^2 be the geometric two-simplex in I^2 having $P_o = (o, o)$, $P_1 = (1, 1)$ and $P_2 = (o, 1)$ as vertices.



Let $H : \Delta_1^2 \to \mathit{H}$ be defined by the formula

$$H(s,\ t) = g_s \circ h_t \qquad \text{for each } (s,\ t) \in \Delta_1^2 \ .$$

Then H is defined on the edges of Δ_1^2 by the following diagram



Certainly H induces a continuous function
$E_2(H) : X \to S(\Delta_1^2\ ,\ X)$.

Identifying Δ_1^2 with the standard simplex Δ^2 and computing

(1) $\partial_2(E_2(H)_* \ \mu_o|_K)$

where K is any compact set in X containing $\operatorname{supp} h \cup \operatorname{supp} g$, we get that (1) is equal to

$$(2)\quad (\delta^* \circ E(g) \circ h_1)_* \mu_o|_K - E(h)_* \mu_o|_K + E(g \circ h)_* \mu_o|_K$$

where $\delta^* : S(\Delta^1, X) \to S(\Delta^1, X)$ is the map induced by the reflection

$$S : \Delta^1 \to \Delta^1$$
$$t \mapsto (1 - t)$$

Hence (1) is equal to

$$(3)\quad (\delta^*)_*(E(g)_* \mu_o|_K) - E(h)_* \mu_o|_K + E(g \cdot h)_* \mu_o|_K$$

It is not difficult to see that the map

$$(4)\quad (\delta^*)_* : C_1 X \to C_1 X$$

induces multiplication by minus one in homology.

This conclues the proof.

□

(J) Remark. Let H be a subgroup of $H_{c,o}(X, \mu_o)$ and let τ be a topology in H such that the inclusion $H_\tau \hookrightarrow H_{c,o}(X, \mu_o)_m$ is continuous.

Then the composite

$$\tilde{H}_\tau \longrightarrow \tilde{H}_{c,o}(X, \mu_o)_m \xrightarrow{\tilde{\theta}} H_1(X, \mathbb{R})$$

is a continuous group homomorphism. In particular, we can apply this remark to the inclusion

$$H_{c,o}(X, \mu_o)_{\underset{\to}{\lim}} \xrightarrow{\text{Id}} H_{c,o}(X, \mu_o)_m$$

and to the inclusion

$$H^{\partial}_{c,o}(M, \mu_o)_m \hookrightarrow H_{c,o}(M, \mu_o)_m$$

in the case where $X = M$ is a manifold with boundary.

7.2 Some Properties of The Mass Flow Homomorphism.

Let X be a locally compact, locally connected, second countable, Hausdorff space, and let $\mu_o \in M_g(X)$. The mass flow homomorphism has some natural properties.

If h is a path of compactly supported homeomorphisms, then the first property will allow us to restrict the domain of h to some (open) subset of X satisfying certain requirements (e.g. compact closure).

The second property describes the behaviour of the mass flow when X is disconnected.

The third property says that if we change μ_o by an equivalent measure then the mass flow is changed by "conjugation".

Finally, we compute an "integral expression" for the mass flow.

Such expression is used to compare the definition of the mass flow homomorphism given above with that given by Fathi in [9].

(A) Let $\mathcal{U}$ be the category of open sets of X ordered by inclusion. Let $U, V \in \mathcal{U}$ such that U is contained in V. Then the inclusion $\zeta : U \hookrightarrow V$ induces a continuous (stratified) group injection

$$\mathcal{H}_{c,o}(U, \mu_o|_U)_m \hookrightarrow \mathcal{H}_{c,o}(V, \mu_o|_V)_m$$

$$h \longmapsto \begin{cases} h & \text{in } U \\ Id & \text{in } V \setminus \operatorname{supp} h \end{cases}$$

which can be lifted to a continuous homomorphism

$$\tilde{\mathcal{H}}_{c,o}(U, \mu_o|_U)_m \xrightarrow{\tilde{\zeta}} \tilde{\mathcal{H}}_{c,o}(V, \mu_o|_V)_m$$

(not injective in general).

The following diagram clearly commutes

$$\begin{array}{ccc} \tilde{\mathcal{H}}_{c,o}(U, \mu_o|_U)_m & \xrightarrow{\tilde{\theta}_U} & H_1(U, \mathbb{R}) \\ \downarrow \tilde{\zeta} & & \downarrow H_1(\zeta) \\ \tilde{\mathcal{H}}_{c,o}(V, \mu_o|_V)_m & \xrightarrow{\tilde{\theta}_V} & H_1(V, \mathbb{R}) \end{array}$$

(B) Let $\{X_i \mid i = o,1,\ldots\}$ be the (denumerable) family of connected components of X and let $\zeta_i : X_i \hookrightarrow X$ be the inclusion map for each i. Then there is a continuous commutative diagram

$$\begin{tikzcd}
{[\check{X}\,\tilde{H}_{c,o}(X_i,\mu_o|X_i)_m]_{\substack{\text{Box}\\ \text{Topology}}}} \arrow[r, "\check{X}\tilde{\theta}_i"] \arrow[d, "V\,\tilde{\zeta}_i", "\wr\|"'] & {[\oplus H_1(X_i,\ \mathbb{R})]_{\substack{\text{Box}\\ \text{Topology}}}} \arrow[d, hook, "VH_1(\zeta_i)"] \\
\tilde{H}_{c,o}(X,\ \mu_o)_m \arrow[r, "\tilde{\theta}"] & H_1(X,\ \mathbb{R})
\end{tikzcd}$$

where

$$\check{X}\,\tilde{\theta}_i(\{[h^i]\}_i) = \{\tilde{\theta}_{X_i}([h^i])\}_i \quad ,$$

and

$$V\,\tilde{\zeta}_i(\{[h^i]\}_i) = \tilde{\zeta}_o([h^o]) \circ \tilde{\zeta}_1([h^1]) \circ \ldots$$

for each $\{[h^i]\}_i$ in the reduced product $\check{X}\,\tilde{H}_{c,o}(X_i,\mu_o|X_i)$. Recall that the reduced product $\check{X}\,\tilde{H}_{c,o}(X_i,\mu_o|X_i)$ is the set of elements in the product such that all of its coordinates are the identity except for possibly finitely many.

The map $V\,\tilde{\zeta}_i$ is a homeomorphism and the linear function $VH_1(\zeta_i)$ is a continuous bijection.

(C) Let f be an arbitrary homeomorphism of X. Then there is a commutative diagram

(B) Let $\{X_i \mid i = o,1,\ldots\}$ be the (denumerable) family of connected components of X and let $\zeta_i : X_i \hookrightarrow X$ be the inclusion map for each i. Then there is a continuous commutative diagram

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\tilde{H}_{c,o}(X,\ \mu_o)_m \arrow[r, "\tilde{\theta}"] & H_1(X,\mathbb{R})
\end{tikzcd}
$$

where

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The map $V\,\tilde{\zeta}_i$ is a homeomorphism and the linear function $VH_1(\zeta_i)$ is a continuous bijection.

(C) Let f be an arbitrary homeomorphism of X. Then there is a commutative diagram

$$\begin{array}{ccc} \tilde{H}_{c,o}(X, \mu_o)_m & \xrightarrow{\tilde{\theta}(X,\, \mu_o)} & H_1(X, \mathbb{R}) \\ \Big\downarrow \widetilde{f(\ )f^{-1}} & & \Big\downarrow H_1(f) \\ \tilde{H}_{c,o}(X, f_*\mu_o) & \xrightarrow{\tilde{\theta}(X, f_*\, \mu_o)} & H_1(X, \mathbb{R}) \end{array}$$

where

$$\widetilde{f(\ )\ f^{-1}}([\, t \mapsto h_t\,]) = [\, t \mapsto f \circ h_t \circ f^{-1}]$$

for each $t \mapsto h_t$ in $P(\tilde{H}_{c,o}(X, \mu_o)_m)$.

Both $\widetilde{f(\ )\ f^{-1}}$ and $H_1(f)$ are isomorphisms (of topological groups and linear spaces respectively). In particular, if f is isotopic to the identity, then $H_1(f)$ is the identity on $H_1(X, \mathbb{R})$.

(D) Let $h \in P(\tilde{H}_{c,o}(X, \mu_o)_m)$ and let $f : X \to K(1, Q)$ be a continuous map, where $K(1, Q)$ is the space defined in example 6.3(2). The induced linear map $H_1(f) : H_1(X, \mathbb{R}) \to H_1(K(1, Q), \mathbb{R}) = \mathbb{R}$ sends $\tilde{\theta}([\, h\,])$ into a real number which can be interpreted (via the isomorphism $\overline{D} : H_1(K(1, Q), \mathbb{R}) \to \mathbb{R}$ of 6.3(2)) as follows:

Let K be a compact subset of X containing the support of h. Then there is an integer ℓ such that f restricts to $f|K : K \to K(1, Q)_\ell$ (in fact K can be any subset of X such that $\operatorname{supp} h \subset K$ and $f(K)$ is contained in some $K(1, Q)_\ell$).

Using h and the projection $\zeta_\ell : K(1, Q)_\ell \to T^1_{\ell+1}$ we can define a continuous homotopy

$$\left[\zeta_\ell \circ (f|_K) \circ h - \zeta_\ell \circ (f|_K)\right] : K \times I \to T^1_{\ell+1}$$

$$(x, t) \mapsto \zeta_\ell \circ f(h_t(x)) - \zeta_\ell \circ f(x)$$

which restricted to $K \times \{o\}$ is identically zero. Therefore, there exists a unique lifting

$$\overline{\zeta_\ell \circ (f\,|_K) \circ h - \zeta_\ell \circ (f|_K)} : K \times I \to \mathbb{R}$$

which restricted to $K \times \{o\}$ is also identically zero.

Assertion.

$$\overline{D} \circ H_1(f)(\tilde{\theta}([h])) = \int_X \overline{\zeta_\ell \circ (f|_K) \circ h_1 - \zeta_\ell \circ (f|_K)}\, d\mu_o$$

(note that the integrand is zero outside supp h)

Proof.

$$\overline{D} \circ H_1(f)(\tilde{\theta}([h])) = \int \lambda \; d(f_{1*} \circ E(h)_* \, \mu_o|_K)$$

$$= \int_K (\lambda \bullet f_1 \circ E(h)|_K)\, d\mu_o|_K$$

$$= \int_K \overline{\zeta_\ell \circ (f|_K) \circ h_1 - \zeta_\ell \circ (f|_K)}\, d\mu_o$$

□

Remark.

If $f : X \to T^1$ is continuous, then

$$\bar{D} \circ H_1(f)\,(\tilde{\theta}([h])) = \int_X \overline{f \circ h_1 - f}\, d\,\mu_o$$

This is exactly the expression A.Fathi uses in [9] to define the mass flow, in the case where X is compact, as a group homomorphism from $\tilde{H}_0(X, \mu_o)_\kappa$ into $\mathrm{Hom}_{\mathbb{Z}}([X, T'], \mathbb{R})$.

7.3 Construction.

The following construction is of fundamental importance for us. In some way, it encompasses the whole algebraic structure of the mass flow homomorphism.

Let X be a locally compact, locally connected, second countable, Hausdorff space and let $\mu_o \in M_g(X)$. Denote by $\mathrm{Ker}\,\tilde{\theta}$ the kernel of the mass flow homomorphism $\tilde{\theta} : \tilde{H}_{c,o}(X, \mu_o)_m \to H_1(X, \mathbb{R})$ and let Π denote the fundamental group $\Pi(H_{c,o}(X, \mu_o)_m)$. Then the diagram

$$\begin{array}{ccccc} & & \Pi & & \\ & & \big\downarrow & & \\ \mathrm{Ker}\,\tilde{\theta} & \hookrightarrow & \tilde{H}_{c,o}(X,\mu_o)_m & \xrightarrow{\tilde{\theta}} & H_1(X, \mathbb{R}) \\ & & \big\downarrow \rho & & \\ & & H_{c,o}(X,\mu_o)_m & & \end{array}$$

induces a commutative diagram with exact columns

$$
(2)\qquad
\begin{tikzcd}
\Pi \cap \operatorname{Ker}\tilde{\theta} \arrow[r, hook] \arrow[d, hook] & \Pi \arrow[r, two heads, "\tilde{\theta}|\Pi"] \arrow[d, hook] & \Gamma_X \arrow[d, hook] \\
\operatorname{Ker}\tilde{\theta} \arrow[r, hook] \arrow[d, "\rho|\operatorname{Ker}\tilde{\theta}"] & \tilde{H}_{c,o}(X, \mu_o)_m \arrow[r, "\tilde{\theta}"] \arrow[d, "\rho"] & H_1(X, \mathbb{R}) \arrow[d] \\
\operatorname{Ker}\theta \arrow[r, hook] & H_{c,o}(X, \mu_o)_m \arrow[r, "\theta"] & H_1(X, \mathbb{R})/\Gamma_X
\end{tikzcd}
$$

where Γ_X is defined to be the image of Π under $\tilde{\theta}$ and $\theta = \theta_X = \theta_{(X,\, \mu_o)}$ is the homomorphism induced by $\tilde{\theta}$ in the quotient. It is easy to see that the equality

$$\text{Kernel}(\ \rho|_{\operatorname{Ker}\tilde{\theta}}\ ) = \text{Kernel}(\ \tilde{\theta}|\Pi\ ) = \Pi \cap \operatorname{Ker}\tilde{\theta} \text{ holds.}$$

Also, the fact that $\tilde{\theta}|\Pi$ is a surjection implies (by a simple diagram chasing) that ρ maps $\operatorname{Ker}\tilde{\theta}$ onto the kernel of θ. This shows that the first column in diagram (2) is exact.

If $H_{c,o}(X, \mu_o)_m$ is locally semi-simply connected, then the map θ is continuous. This follows immediately from the continuity of $\tilde{\theta}$ and the fact that in this case ρ is a covering map.

The next paragraph intends to provide a geometric description of the restriction of the mass flow homomorphism to the fundamental group Π.

7.4 <u>Construction.</u>

Let X be a <u>connected,</u> locally path connected, locally compact, Hausdorff space. Observe that if a space is locally connected, locally compact, second countable and Hausdorff, which is what we have usually been assuming, then such a space is automatically locally path connected (see Hocking and Young [12]).

If x_0 is a point in X, then the (continuous) function ev_0, from the group of homeomorphisms isotopic to the identity into X, given by evaluation at x_0,

$$ev_0 : H_0(X)_\kappa \to X$$
$$h \mapsto h(x_0) ,$$

induces a group homomorphism

$$\Pi(ev_0) : \Pi(H_0(X)_\kappa) \to \Pi(X, x_0)$$

which assigns to each class $[\gamma]$ of a loop γ in $\Omega(H_0(X)_\kappa)$ the class $[E(\gamma)x_0]$ of the loop $t \mapsto \gamma_t(x_0)$ $(o \le t \le 1)$.

Since $\Pi(H_0(X)_\kappa)$ is abelian (consider the homotopy $(s,t) \mapsto \gamma_1(s) \circ \gamma_2(t)$ $(o \le s,t \le 1)$), the next assertion is not unnatural.

(A) <u>Assertion</u>. If h is a loop of homeomorphisms in $H_0(X)_\kappa$ (based at the identity), then $\Pi(ev_0)[h]$ is in the group theoretic centre $\mathbf{Z}(\Pi(X, x_0))$ of the fundamental group of X (i.e. $\Pi(ev_0)[h]$ commutes with every element in $\Pi(X, x_0)$).

<u>Proof.</u>

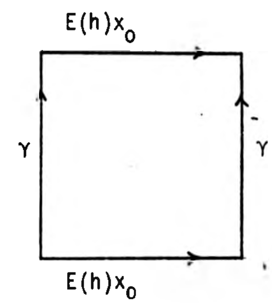
Let $\gamma \in \Omega(X, x_0)$ be a loop in X. Then $G : I \times I \to X$ given by $G(s,t) = h_s(\gamma(t))$ for all $(s,t) \in I \times I$, is such that

$$G(s,o) = E(h)\, x_0$$

$$G(s,1) = E(h)\, x_0$$

$$G(o,t) = \gamma(t)$$

$$G(1,t) = \gamma(t)$$



This proves that $\Pi(ev_0)\,[h]$ commutes with $[\,\gamma\,]$ in $\Pi(X,x_0)$.

Note that G can be thought as a map from the torus $S^1 \times S^1$ into X.

□

(B) <u>Remarks.</u> $\mathbb{Z}(\,\Pi(X, x_0))$ is an invariant of X that does not depend of the base point x_0, for if $x_1 \in X$ then any two curves from x_0 to x_1, say γ_0 and γ_1 define the same isomorphism from $\mathbb{Z}(\,\Pi(X, x_0))$ onto $\mathbb{Z}(\Pi(X, x_1))$ (i.e. $[\,\gamma_0^{-1}\,\zeta\,\gamma_0\,] = [\gamma_1^{-1}\,\zeta\,\gamma_1\,]$ for all $\zeta \in \Omega(X, x_0)$ such that $[\zeta] \in \mathbb{Z}(\Pi(X, x_0))$).

Hence $\mathbb{Z}(\Pi(X))$ is well defined. If $h \in \Omega(H_0(X)_K)$ then these remarks and the homotopy $(s,t) \mapsto h_s(\gamma_0\,(t))$ show that $E(h)x_0$ and $E(h)x_1$ define the same element in $\mathbb{Z}(\Pi(X))$.

If $(Y, \{y_0\})$ is a topological pointed space and $f : (Y, y_0) \to (H_0(X)_\kappa, \mathrm{Id})$ a continuous function, then the composite $Y \xrightarrow{f} H_0(X)_\kappa \xrightarrow{ev_0} X$ induces a group homomorphism

$$\Pi(Y, y_0) \to \mathbb{Z}(\Pi(X))$$

$$[\gamma] \mapsto [E(f \circ \gamma)x_0].$$

In particular, we can apply this remark to the inclusion $H_{c,o}(X, \mu_0)_\tau \hookrightarrow H_0(X)_\kappa$ ($\tau = m$ or $\varinjlim$), where $\mu_0 \in M_g(X)$ to obtain a group homomorphism

$$\Pi(ev_0) : \Pi(H_{c,o}(X, \mu_0)_\tau) \to \mathbb{Z}(\Pi(X)).$$

Observe that if X is not compact then this homomorphism is trivial, for if $h \in \Omega(H_{c,o}(X, \mu_0)_\tau)$ then some point, say $x \in X$, must remain fixed, that is, $E(h)x$ must be the constant loop $t \mapsto x$ $(0 \leq t \leq 1)$. The same is true for the case where $X = M$ is a manifold with non-empty boundary and we apply the above remark to the inclusion $H^\partial_{c,o}(M, \mu_0)_\tau \hookrightarrow H_0(M)_\kappa$: The homomorphism $\Pi(ev_0) : \Pi(H^\partial_{c,o}(M, \mu_0)_\tau) \to \mathbb{Z}(\Pi(M))$ is again trivial.

The next proposition says that the following diagram commutes (up to scalar multiplication by $\mu_0(M)$ in the case where M is compact)

$$\begin{array}{ccc}
\Pi(\mathcal{H}_{c,o}(M,\mu_o)_m) & \xrightarrow{\Pi(ev_o)} & \mathbb{Z}(\Pi(M)) \\
\Big\downarrow \eta & & \Big\downarrow \mathrm{Hwz} \\
\tilde{\mathcal{H}}_{c,o}(M,\mu_o)_m & \xrightarrow{\tilde{\theta}} & H_1(M,\mathbb{R})
\end{array}$$

where Hwz : $\Pi(M, x_0) \to H_1(M, \mathbb{R})$ is the map that sends the homotopy class of a loop γ on M at x_0 into the homology class of the atomic probability on $S(\Delta^1, M)_K$ concentrated at $\{\gamma\}$.

7.5 Proposition.

Let M be a connected, second countable manifold, let $\mu_0 \in \mathcal{M}_g^{\partial}(M)$ and let x_0 be any point in M. Let h be a loop on $\mathcal{H}_{c,o}(M, \mu_o)_m$ based at the identity. If M is compact, then $\tilde{\theta}([h])$ is equal to $(\mu_0(M))\mathrm{Hwz}([E(h)x_0])$, where $\mathrm{Hwz}([E(h)x_0])$ is the homology class of the atomic probability concentrated at the loop $s \mapsto h_s(x_0)$ $(o \le s \le 1)$. If M is non-compact, then $\tilde{\theta}([h])$ is zero.

Proof.

Proposition 1 and Lemma 7 in Berlanga and Epstein [3] show that there is a continous function $\varphi : I^n \to M$ and a Radon measure ν_0 on I^n such that

(1) $K \subset \operatorname{supp} h \subset \varphi(I^n)$

(2) $\varphi_* \nu_0 = \mu_0 | \varphi(I^n)$

Since the point $x_o \in M$ is irrelevant we can assume that $\varphi(o, o, \ldots, o) = x_o$.

Comment. After neglecting $M \setminus \varphi(I^n)$, we have cut $\varphi(I^n)$ along the "scar" $\varphi(\partial I^n)$ to get a cube. The composite

$$E(h) \circ \varphi : I^n \to S(\Delta^1, M)$$

may be interpreted as a (singular) cube of circles in $S(\Delta^1, M)$ supporting the measure μ_o(appropriately restricted). Because I^n is contractible, this cube of circles, say C, can be homotoped uniformly to the single circle $E(h)x_o$. As C is being contracted, the mass it carries must concentrate around $E(h)x_o$. Now we continue the formal proof.

Let $c : I^n \times I \to I^n$ be the contraction of I^n defined by

$$c(x,t) = (1-t)x \qquad \text{for all } (x,t) \in I^n \times I.$$

Let $H : I \times I \times I^n \to M$ be defined by the formula

$$H(s,t)\,x = h_s(\varphi(c(x,t)))$$

$$\text{for all } (s,t) \in I \times I \quad \text{and} \quad x \in I^n .$$

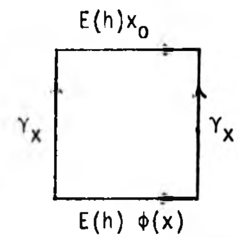
Then,

$$H(s,o)x = h_s(\varphi(x))$$

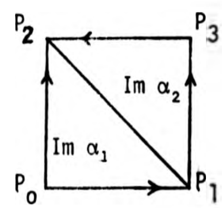
$$H(s,1)x = h_s(x_o)$$

$$H(o,t)x = H(1,t)x = \varphi(c(x,t)) = \gamma_x(t)$$

for all $(s,t) \in I \times I$ and $x \in I^n$.



Define the affine simplices $\alpha_j: \Delta^2 \to I \times I$ $(j=1,2)$ where $\alpha_1 = (P_0, P_1, P_2)$, $\alpha_2 = (P_1, P_3, P_2)$ and $P_0 = e_0$, $P_1 = e_1$, $P_2 = e_2$, $P_3 = e_1 + e_2$ are the vertices of the unit square.



Define, for $j = 1,2$

$$E_2(H \circ \alpha_j) : I^n \to S(\Delta^2, M)$$

$$x \mapsto ((s,t) \mapsto H(\alpha_j(s,t))x).$$

Let $\delta : \Delta^1 \to \Delta^1$ be such that $\delta(t) = (1-t)$ for each $t \in \Delta^1$, (see 7.1 I(4)). Then

$$\partial_2(E_2(H \circ \alpha_1)_* \nu_o + E_2(H \circ \alpha_2)_* \nu_o)$$

$$=(E(h) \circ \varphi)_* \nu_o + (\delta^* \circ (x \mapsto E(h)x_o))_* \nu_o$$

$$= E(h)_*(\varphi_* \nu_o) + (\delta^*)_*(\nu_o(I^n)\{E(h)x_o\})$$

$$= E(h)_*(\mu_o|_{\varphi(I^n)}) + (\delta^*)_*(\nu_o(I^n)\{E(h)x_o\})$$

where $\{E(h)x_o\}$ represents the atomic probability supported at $E(h)x_o$.

Hence,

$$\tilde{\theta}([h]) = \mu_o(\varphi(I^n))\,[\{E(h)x_o\}] .$$

In particular, if $\operatorname{supp} h \neq M$ then $E(h)x : s \mapsto h_s(x)$ $(o \leq j \leq 1)$ is a constant loop for some $x \in M$. Therefore $E(h)x_o$ is homologically trivial for it is homotopic to $E(h)x$ (see 7.4). This is always the case when M is non-compact. If M is compact and $\operatorname{supp} h = M = \varphi(I^n)$ then

$$\tilde{\theta}([h]) = \mu_o(M)\,[\{E(h)x_o\}] .$$

□

7.6 Remark.

If A is a contractible space, ν_o' a Borel measure on A and $\varphi' : A \to M$ a continuous function such that $\varphi'_* \nu_o' = \mu_o|_K$ where K is some compact set containing $\operatorname{supp} h$, then the above proof holds if we substitute I^n, ν_o and φ by A, ν_o' and φ' respectively.

If X is a connected, locally compact, locally contractible, second countable, Hausdorff space ; x_0 a point in X and μ_1 a Borel measure on X, then it is always possible to construct a Borel measure ν_1 on the space of paths $P(X,x_0)_K$ (which is contractible via the homotopy $(\gamma, s) \mapsto (t \mapsto \gamma(st)))$ such that

$\rho_* \nu_1 = \mu_1$, where $\rho : P(X, x_0) \twoheadrightarrow X$ is the projection $\gamma \mapsto \gamma(1)$ (this follows from the fact that contractible neighbourhoods in X may be lifted continuously to $P(X, x_0)_K$ so we can construct ν_1 supported in a lifting of some suitable covering of X).

Hence the above proposition is extended to this class of spaces. In particular, every connected, locally compact, metric ANR lies in this category. This follows from the facts that a metric ANR is locally contractible (see Hu [13 , p. 32] or Spanier [23 p. 37, exercise 6]) and that every connected, locally compact metric space is second countable (see Spivak [24 , Theorem 2, p.1.5]).

7.7 Lemma.

Let X be a locally compact, separable metric ANR, and let $\mu_0 \in M_g(X)$. Consider the diagram

$$\begin{array}{ccccc}
\operatorname{Ker} \tilde{\theta} & \hookrightarrow & \tilde{H}_{c,o}(X, \mu_0)_m & \xrightarrow{\tilde{\theta}} & H_1(X, \mathbb{R}) \\
\downarrow{\scriptstyle \rho|\operatorname{Ker}\tilde{\theta}} & & \downarrow{\scriptstyle \rho} & & \downarrow \\
\operatorname{Ker} \theta & \hookrightarrow & H_{c,o}(X, \mu_0)_m & \xrightarrow{\theta} & H_1(X, \mathbb{R})/\Gamma_X
\end{array}$$

defined in paragraph 7.3 .

If ρ is a covering projection, then $\rho|\mathrm{Ker}\,\tilde{\theta}$ is also a covering map.

Proof.

By 7.2(B) and the fact that an arbitrary (reduced) product of covering spaces in the box topology is a covering space, we can assume, without loss of generality, that X is connected.

We now consider two cases.

Case 1, in which X is non-compact. In this case $\tilde{\Gamma}_X$ is trivial by 7.5 and 7.6, so $\mathrm{Ker}\,\tilde{\theta}$ is the full inverse image of $\mathrm{Ker}\,\theta$ under ρ. Hence the result follows.

Case 2, in which X is compact. Assume, without loss of generality, that $\mu_0(X) = 1$. Therefore, 7.5 and 7.6 imply that Γ_X is contained in the integral part of the finite dimensional space $H_1(X, \mathbb{R})$ (a compact metric ANR is of the same homotopy type of a compact polyhedron, hence it has finitely generated integral singular homology (see Hu [13, Chapter I])). Hence Γ_X is discrete in the (natural) euclidean topology of $H_1(X, \mathbb{R})$.

Choose, by the continuity of the mass flow homomorphism, a neighbourhood $\mathcal{U}$ of the identity in $\tilde{H}_0(X, \mu_0)_K$ with $\tilde{\theta}(\mathcal{U}) \cap \Gamma_X = \{o\}$. Then it follows that $\rho(\mathrm{Ker}\,\tilde{\theta} \cap \mathcal{U}) = \mathrm{Ker}\,\theta \cap \rho(\mathcal{U})$, proving that $\rho|\mathrm{Ker}\,\tilde{\theta} : \mathrm{Ker}\,\tilde{\theta} \twoheadrightarrow \mathrm{Ker}\,\theta$ is an open map. Hence $\mathrm{Ker}\,\theta$ is the quotient space obtained from $\mathrm{Ker}\,\tilde{\theta}$ under the action of the discrete group $\Pi(H_0(X, \mu_0)_K) \cap \mathrm{Ker}\,\tilde{\theta}$. Thus $\rho\,|\mathrm{Ker}\,\tilde{\theta}$ must be a covering projection.

□

§8. The kernel of the mass flow homomorphism in the case of a σ-compact manifold.

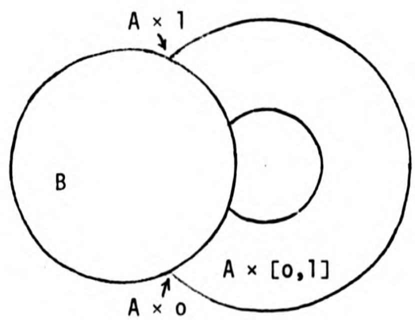
8.1 Why $\tilde{\theta}$ is called the mass flow homomorphism?

The following discussion is essentially taken from Fathi [9 , p.p. 73-74]:

Let A and B be locally connected, second countable, Hausdorff spaces. Assume further that A is compact and that B is locally compact. Now let $\phi: A \times \{o,1\} \to B$ be some embedding.

We define X by glueing $A \times [o,1]$ to B using the map ϕ

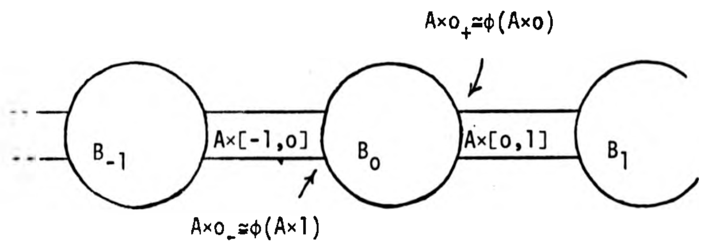
$$X = (A \times [o,1] \sqcup B)/\phi(a,o) \sim (a,o),\ \phi(a,1) \sim (a,1).$$



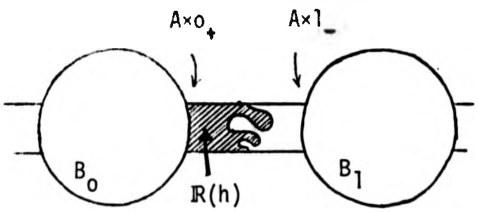
We can define a natural map $f: X \to T^1$ by:

$$f(x)=\begin{cases} t \bmod 1, & \text{if } x=(a,t) \in A \times [o,1], \\ o, & \text{if } x \in B. \end{cases}$$

We obtain a covering $\bar{X} \to X$ as the pull-back by f of the covering $R \to T^1$. The space X can also be defined as in the figure below:



Suppose that μ is a measure on X, such that $\mu|_{A \times [o,1]} = \nu \times dt$, where ν is a good measure on A and dt is Lebesgue measure on $[o,1]$. We can "lift" μ to a measure $\bar{\mu}$ in $\bar{X}$. Let h be a measure preserving, compactly supported isotopy in $IS_c(X, \mu)$. Then we can lift h in a unique way to $h \in IS(X, \mu)$. Observe that supp $\bar{h}$ is not compact if $h_t \neq Id$ for some $t \in I$. Moreover, $\bar{h}$ depends continuously on h (in the compact open topology) and $\bar{h}$ commutes with the covering transformations of $\bar{X} \to X$. Suppose that h is close enough to the identity, then we can define a region $\mathbb{R}(h) \subset \bar{X}$ which consists of the points between $A \times 0_+$ and $\bar{h}_1(A \times 1/2)$.



Assertion:

If $H_1(T^1, \mathbb{R})$ is identified with IR via the isomorphism found in example 6.3(1), then

$$H_1(f)(\tilde{\theta}([h])) =$$

$$\bar{\mu}(\mathbb{R}(h)) - \bar{\mu}(A \times [o, 1/2])$$

We omit the proof of this fact which is based on an ingenious integration over a fundamental domain of the covering $\bar{X} \to X$ (see Fathi [9 , Proposition 5.4, p.74]).

Remark (1).

The above assertion can be interpreted by saying that $\tilde{\theta}([h])$ is the mass that has passed algebraically through the "membrane" $A \times 1/2 \subset X$. If we imagine $A \times [o,1]$ as a pipe, this explains the name of the mass flow homomorphism.

Remark (2)

Let $a,b \in \langle o,1\rangle$. Suppose that A is a connected manifold V^{n-1} , and that $\mu|_{V^{n-1} \times [o,1]} = \nu \times dt$, where ν is a good measure on V^{n-1} . If we define $C_o(h)$ as the connected component of $V^{n-1} \times [o,1] \setminus h_1(V^{n-1} \times [a,b])$ which contains $V^n \times \{o\}$, then we have, by the above assertion, that

$$(1) \quad H_1(f)(\tilde{\theta}([h])) = \mu(C_o(h)) - \mu(V^{n-1} \times [o,a\rangle)$$

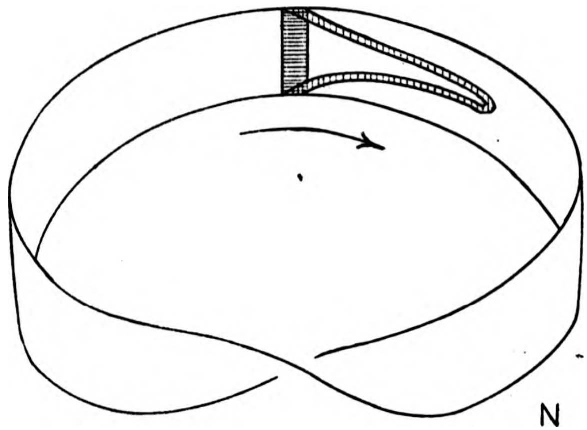
In particular, if the space $X = (V \times I) \cup_\phi B$ is such that $\phi(V \times \{o\})$ and $\phi(V \times \{1\})$ lie in different connected components of B, then $f : X \to T^1$ is homotopically trivial. Hence we are just asserting that

$$(2) \quad \mu(C_o(h)) = \mu(V \times [o,a\rangle).$$

8.2 Example

In Proposition 8.4 below we are going to prove that the mass flow homomorphism is surjective for a manifold M.

Suppose for a moment that $N \subset M$ is a (possibly non-orientable) compact tubular neighbourhood of some circle σ embedded in M and representing a non-trivial element in $H_1(M, \mathbb{R})$.



Using 8.1, A.Fathi showed how to circulate mass inside N in order to produce homeomorphisms with non-zero mass flow along σ. We want to do the same, but without the aid of tubular neighbourhoods.

Let A be the unit $(n-1)$-cube I^{n-1}, let B be an n-dimensional connected manifold with boundary, and let $\phi : A \times \{0,1\} \to \partial B$ be an embedding which extends to an (open) embedding of $\mathbb{R}^{n-1} \times \{0,1\}$ into ∂B. Then $X = A \times [0,1] \cup_{\phi} B$ is a manifold whose boundary is

$$\partial X = \partial A \times I \cup (\partial B \setminus \phi(\mathring{A} \times \{0,1\}))$$

By definition, X is the result of adding an n-<u>handle of index</u> 1 <u>to</u> B.

Suppose now that μ is a measure in $M_g^{\partial}(X)$ such that $\mu|_{A \times I} = (\alpha_0 m) \times dt$, where m is Lebesgue measure on A and $\alpha_0 > 0$.

We want to show that the mass flow homomorphism $\tilde{\theta}_X$ is not trivial.

Let $f : X \to T^1 = \mathbb{R}/\mathbb{Z}$ be given by

$$f(x) = \begin{cases} u \bmod 1 & \text{if } x = (a, u) \in A \times I \\ o & \text{if } x \in B . \end{cases}$$

Then it is not difficult to see that $H_1(X, \mathbb{R}) \cong H_1(B, \mathbb{R}) \oplus \mathbb{R}$ (by the Mayer-Vietoris Theorem), and that the linear map $H_1(f) : H_1(X, \mathbb{R}) \to H_1(T^1, \mathbb{R})$ may be interpreted as the projection of $H_1(B, \mathbb{R}) \oplus \mathbb{R}$ onto its second factor.

Choose some continuous function $\delta : A \to [o, 1/3]$ such that $\alpha_o \int_A \delta dm = \alpha_o/4$ and $\delta \big|_{\partial A} = o$.

Define, for each $t \in I$, the embedding

$$A \times [1/4, 1/2] \hookrightarrow X$$

$$(a, u) \mapsto (a, u + t\,\delta(a)).$$

Then the extension of isotopies theorem 5.6 says that we can find a path $h : I \to H^{\partial}_{c,o}(X, \mu)_{\underset{\to}{\lim}}$ such that $h_t(a, u) = (a, u + t\,\delta(a))$ for each $(a, u) \in A \times [1/4, 1/2]$.

By 8.1, it follows that $\tilde{\theta}_X([h]) = (\beta, \alpha_o/4)$ in $H_1(B, \mathbb{R}) \oplus \mathbb{R}$.

Furthermore, by adding an extra parameter $s \in [o, \alpha_o/4]$, we can construct a continuous map $\gamma : [o, \alpha_o/4] \to \tilde{H}^{\partial}_{c,o}(X, \mu)_{\underset{\to}{\lim}}$

(e.g. $\gamma(s)_t = [h_{4st/\alpha_o}]$ for each $s \in [o, \alpha_o/4]$ and each $t \in I$) with the property that, for each $s \in [o, \alpha_o/4]$, $\tilde{\theta}_X(\gamma(s)) = (\beta_s, s)$ in $H_1(B, \mathbb{R}) \oplus \mathbb{R}$.

Finally, it is easy to extend γ to a continuous map, denoted again by γ, from all of $\mathbb{R}$ into $\tilde{H}^{\partial}_{c,o}(X, \mu)_{\underrightarrow{\lim}}$ satisfying the equality $\tilde{\theta}_X(\gamma(s)) = (\beta_s, s)$ for each $s \in \mathbb{R}$.

Before we can state Proposition 8.4, we must change the topology of the first homology vector space.

8.3 <u>Definition.</u>

Let W be a real vector space. Then the $\underrightarrow{\lim}$-topology on W is the topology coinduced by the coherent family

$$S = \{E \subset W \mid E \text{ is a finite dimensional } \mathbb{R}\text{-linear space}\}$$

where each $E \in S$ is given its standard euclidean topology.

If the dimension of W is at most countable, then $W_{\underrightarrow{\lim}}$ is a locally convex vector space. Otherwise the (affine) space $W_{\underrightarrow{\lim}}$ is not a topological vector space (see Dugundji [6]).

Let M be a second countable manifold and let $\mu_o \in M^{\partial}_g(M)$.

Recall that, in this case, the mass flow homomorphism $\tilde{\theta}_M : \tilde{H}_{c,o}(M, \mu_o)_m \to H_1(M, \mathbb{R})$ is a continuous group homomorphism and that $H_1(M, \mathbb{R})$ is Hausdorff (see 6.5 and 7.1(H)).

Using the fact that the projection $\tilde{H}_{c,o}(M,\mu_o)_m \to H_{c,o}(M,\mu_o)_m$ is a covering map and Lemma 4.3, it is not difficult to see that if $F \subset \tilde{H}_{c,o}(M, \mu_o)_m$ is compact, then there is a compact $K \subset M$ such that $\tilde{\theta}_M(F)$ is contained in the (finite dimensional) image of the canonical map $H_1(K, \mathbb{R}) \to H_1(M, \mathbb{R})$.

Therefore, if $H_1(M, \mathbb{R})_{\underrightarrow{\lim}}$ denotes the real vector space $H_1(M, \mathbb{R})$ with the $\underrightarrow{\lim}$-topology, then the mass flow homomorphism as a function from $k(\tilde{H}_{c,o}(M, \mu_o)_m) = \tilde{H}_{c,o}(M, \mu_o)_{\underrightarrow{\lim}}$ into $H_1(M, \mu_o)_{\underrightarrow{\lim}}$ is continuous (see the diagram in 6.1 (B)).

8.4 <u>Proposition.</u>

Let M be a second countable manifold and let $\mu_o \in M_g^{\partial}(M)$. Then the map

$$\tilde{\theta} : \tilde{H}_{c,o}(M, \mu_o)_{\underrightarrow{\lim}} \to H_1(M, \mathbb{R})_{\underrightarrow{\lim}}$$

is surjective.

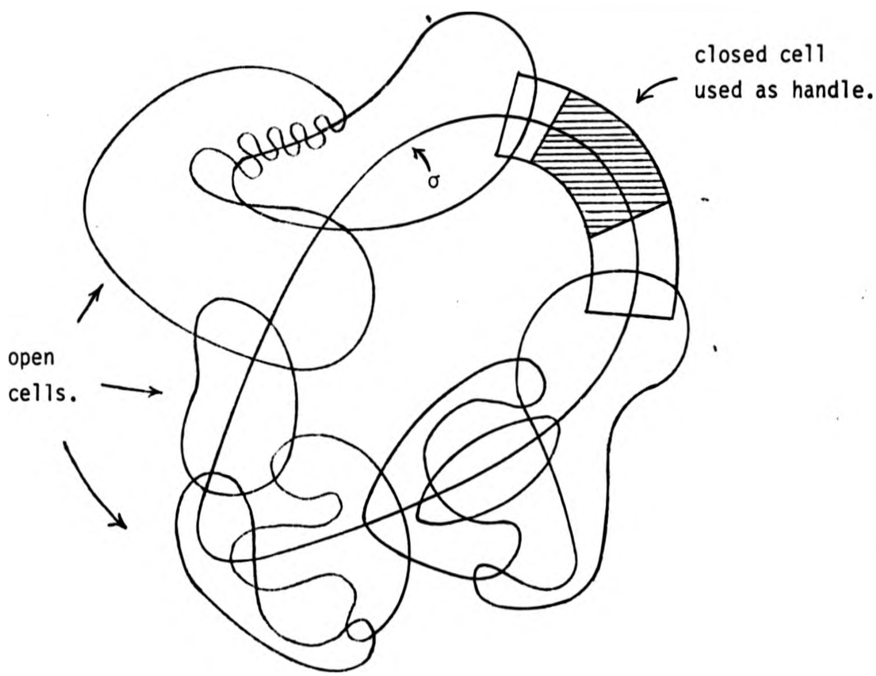
Furthermore, $\tilde{\theta}$ has a continuous section when the domain and codomain of $\tilde{\theta}$ are given their respective $\underrightarrow{\lim}$-topologies.

<u>Proof.</u>

Let M be of dimension n.

Let z be a non-trivial element in $H_1^s(M, \mathbb{Z})$ for which there exists an embedding $\sigma : T^1 \hookrightarrow \operatorname{Int} M$ such that σ represents z.

By covering the image of σ with n-cells, we can construct a neighbourhood X of $\sigma(T^1)$ in the interior of M such that X can be expressed as the result of adding an n-handle of index 1 to a submanifold of Int M of dimension n. (see figure below).



Therefore, X = (closed n-cell) $\cup$ B is homeomorphic to some $I^{n-1} \times I \cup_{\phi} B$ as defined in example 8.2 .

We can assume, by the von Neumann-Oxtoby-Ulam theorem, that the measure μ_0 restricted to X is taken under this homeomorphism to a measure such as that in 8.2 .

Since T^1 is an ANR, we can take X small enough so that the canonical map $H_1^s(B, \mathbb{Z}) \to H_1^s(M, \mathbb{Z})$ is trivial.

Using example 8.2 and the following commutative diagram

$$\begin{array}{ccccc}
\tilde{H}^{\partial}_{c,o}(X, \mu_o|_X) & \xrightarrow{\tilde{\theta}_X} & H_1(X, \mathbb{R}) = H_1(B, \mathbb{R}) \oplus \mathbb{R}\, z & & \\
\downarrow & & \downarrow & \downarrow & \text{Projection on the 2nd factor.} \\
\tilde{H}^{\partial}_{c,o}(M, \mu_o) & \xrightarrow{\tilde{\theta}_M} & H_1(M, \mathbb{R}) & \hookleftarrow & \mathbb{R}\, z
\end{array}$$

we can find a continuous

$$\gamma_\sigma : \mathbb{R}z \to \tilde{H}^{\partial}_{c,o}(M, \mu_o)_{\underrightarrow{\lim}}$$

such that $\tilde{\theta}_M \circ \gamma_\sigma$ is the identity on $\mathbb{R}\, z$.

Now choose an at most countable family $\{\sigma : T^1 \hookrightarrow M\}$ of continuous embeddings representing a base for $H_1(M, \mathbb{R})$.

Note that any element in $H_1(M, \mathbb{R})$ is a <u>finite</u> linear combination of basic vectors. Hence, we can multiply the corresponding family of sections

$$\{ \gamma_\sigma : \mathbb{R}[\sigma] \to \tilde{H}^{\partial}_{c,o}(M, \mu_o)_{\underrightarrow{\lim}} \}$$

in some fixed order (for $\gamma_{\sigma_2} \circ \gamma_{\sigma_1} \neq \gamma_{\sigma_1} \circ \gamma_{\sigma_2}$ in general) to obtain a continuous section

$$\gamma : H_1(M, \mathbb{R})_{\underrightarrow{\lim}} \to \tilde{H}^{\partial}_{c,o}(M, \mu_o)_{\underrightarrow{\lim}}$$

This concludes the proof. □

Digression.

Certainly the section γ just constructed has no respect for the algebraic structures involved.

If the dimension of M is greater than two, then it is possible to choose a locally finite family, $\{\sigma : T^1 \hookrightarrow M\}$ of continuous disjoint embeddings representing a base for $H_1(M, \mathbb{R})$.

If it happened that each $\sigma : T^1 \hookrightarrow M$ had a tubular neighbourhood, then we could construct a section which would be also a group homomorphism.

8.5 Corollary.

Let M be a second countable manifold and let $\mu_o \in M_g^\partial(M)$. Then there exists a homeomorphism

$$\tilde{H}_{c,o}(M, \mu_o)_{\underrightarrow{\lim}} \cong k\,[(\ker \tilde{\theta}) \times H_1(M, \mathbb{R})_{\underrightarrow{\lim}}]$$

where $\tilde{H}_{c,o}(M, \mu_o)$ is given the $\underrightarrow{\lim}$-topology, the kernel of the mass flow homomorphism $\operatorname{Ker} \tilde{\theta}$ is given its subspace topology, and the right hand side space of the equation is the compactly generated space associated with the product $(\operatorname{Ker}\tilde{\theta}) \times H_1(M, \mathbb{R})_{\underrightarrow{\lim}}$.

In particular $\operatorname{Ker} \tilde{\theta}$ is connected and locally contractible.

Proof.

The existence of a homeomorphism follows from the existence of a continuous section. The fact that $H_{c,o}(M, \mu_o)_{\underrightarrow{\lim}}$ is connected and locally contractible implies that its universal covering has these same properties. Therefore, the last statement of the corollary follows.

□

8.6 Corollary.

Under the same hypothesis as above, $\text{Ker}\,\theta$ ($\subset \mathcal{H}_{c,o}(M, \mu_o)_{\underset{\to}{\lim}}$) is connected and locally contractible.

Proof.

This is an easy consequence of Lemma 7.7 .

□

8.7 Remarks.

It also follows from Corollary 8.6 that $\text{Ker}\,\tilde{\theta}$ is simply connected. Hence, $\text{Ker}\,\tilde{\theta}$ is the universal covering space of $\text{Ker}\,\theta$.

It is not difficult to see that if we endow $\text{Ker}\,\tilde{\theta}$ and $\text{Ker}\,\theta$ with the Whitney topology m, then $[\,\text{Ker}\,\tilde{\theta}\,]_m$ is also the universal covering space of $[\,\text{Ker}\,\theta\,]_m$. In particular, $[\,\text{Ker}\,\theta]_m$ is connected and locally path connected.

Observe that 8.4, 8.5, 8.6 and the remarks just made hold if we substitute $\mathcal{H}^{\partial}_{c,o}(M,\mu_o)$ for $\mathcal{H}_{c,o}(M, \mu_o)$.

Now, we turn ourselves to the study of the algebraic structure of the kernel of the mass flow homomorphism.

8.8 Lemma

Let $\nu_o \in M^{\partial}_g(I^n)$. Then $\mathcal{H}^{\partial}(I^n, \nu_o)$ is contractible.

□

This is a consequence of the so called "Alexander's trick" and the von Neumann-Oxtoby-Ulam theorem (see Fathi [9 , Proposition 3.8, p.59]).

8.9 Lemma.

Let M be a second countable manifold, let K be a closed n-cell contained in M and let $\mu_o \in M_g^{\partial}(M)$. Suppose $h \in H(M, \mu_o)$ is isotopic to the identity by a μ_o-preserving isotopy having its support in K. Then $\theta(h) = o$. If $h \in H(M, \mu_o)$ has its support in $\mathring{K}$, then h is isotopic to the identity in $\mathring{K}$, and $\theta(h) = o$.

Proof.

Any homeomorphism in $H(K,\mu_o|_K)$ which fixes $\mathrm{Fr}\,K$ pointwise can be extended to a homeomorphism in $H(M, \mu_o)$, by defining such an extension to be the identity out of K (note that $\mathrm{Fr}\,K$ may be different than ∂K).

Let $H^{Fr}(K, \mu_o|_K)$ denote the group of homeomorphisms of K preserving $\mu_o|_K$ and fixing $\mathrm{Fr}\,K$ pointwise endowed with the compact open topology.

Let $H_o^{Fr}(K, \mu_o|_K)$ be the path component of the identity in $H^{Fr}(K, \mu_o|_K)$. Then it is not difficult to see that there is a commutative diagram

$$\begin{array}{ccc} H_o^{Fr}(K, \mu_o|_K) & \xrightarrow{\theta} & H_1(K, \mathbb{R}) \equiv o \\ \downarrow & & \downarrow \\ H_{c,o}(M, \mu_o) & \xrightarrow{\theta} & H_1(M, \mathbb{R})/\Gamma_M \end{array} .$$

This implies the first part of the lemma. The rest of it is a consequence of 8.8 .

□

8.10 Proposition.

Let M^n be a second countable, n-dimensional manifold and let $\mu_o \in M_g^\partial(M)$. If $h \in H_{c,o}(M, \mu_o)$ is in Ker θ_M, then h can be written as a composition $h = h_1 \circ h_2 \circ \ldots \circ h_q$ such that, for each i, h_i is in $H_{c,o}(M, \mu_o)$ and supp h_i is contained in an n-cell.

Furthermore, if $h \in H^\partial_{c,o}(M, \mu_o)$, then each h_i can be assumed to fix ∂M pointwise.

We first prove one lemma.

8.11 Lemma.

Let N be an n-dimensional manifold and let S be a compact, connected, bicollared (n-1)-submanifold of N (see Definitions 2.1). Let $\mu \in M_g^\partial(N)$ such that

(1) $\mu|_{S \times <-1, 2>} = \nu \times dt$, where $S \times <-1, 2>$ is a bicollar of $S = S \times \{1/2\}$, ν is a good measure on S and dt is Lebesgue measure on $<-1,2>$.

Let $h : N \times I \to N$ be a compactly supported isotopy preserving μ satisfying

(2) $\theta_N(h_t) = 0$ for each $t \in I$;

(3) $h_t(S \times [1/4, 3/4]) \subset S \times \langle o, 1\rangle$ for each $t \in I$.

Then there is an isotopy $b : M \times I \to M$ preserving μ such that

(4) $\operatorname{supp} b \subset S \times \langle o, 1\rangle$;

(5) $b_t \big| S \times [1/4, 3/4] = h_t' \big| S \times [1/4, 3/4]$ for each $t \in I$.

Furthermore, if D' is a closed set in ∂N and W a neighbourhood of D' in ∂N such that $h_t|_W = Id_W$, for each $t \in I$, then b can be chosen in such a way that $b_t|_{D'} = Id_{D'}$, for each $t \in I$.

Proof.

Define, for each $t \in I$, $C_{o,t}$ (resp. $C_{1,t}$) as the connected component of $S \times [o, 1] \setminus h_t(S \times [1/4, 3/4])$ which contains $S \times \{o\}$ (resp. $S \times \{1\}$).

By theorem 5.6 and Remark 5.7, it is enough to prove that, for each $i = 1, 2$,

(6) $\mu(C_{i,t}) = \mu(C_{i,o})$ if $t \in I$. But now (6) follows from Remark (2) at the end of 8.1 and the assumption that h is an isotopy in $\operatorname{Ker} \theta$.

□

Proof of Proposition 8.10 .

Let E^n denote either $\mathbb{R}^n$ or the upper half space $\mathbb{H}^n = \mathbb{R}^{n-1} \times [o, \infty\rangle$, and let $B_o(r)$ denote the standard euclidean closed ball of radius r with centre at the origin.

Let U be an open subset of M with compact closure satisfying $h \in H_{c,o}(U, \mu_o|_U)$ and $\theta_U(h) = o$. Since $\mathrm{Cl}\, U$ is compact, we can find a finite number of coordinate charts, say $\phi_j : E^n \hookrightarrow M^n$ $(i \le j \le \ell)$, such that $\cup \{ \phi_j(B_o(1) \cap E^n) \mid 1 \le j \le \ell \}$ is an open set containing $\mathrm{Cl}\, U$.

Therefore, by induction on the number ℓ of charts required to cover $\mathrm{Cl}\, U$, it is enough to prove that, for a given coordinate chart $\phi : E^n \hookrightarrow M^n$, any sufficiently small h in $\mathrm{Ker}\, \theta_U$ can be written as a product $g \circ f$ such that

(1) $g \in H_{c,o}(\overset{\circ}{\widehat{2A}}, \mu_o|_{\overset{\circ}{\widehat{2A}}})$;

(2) $f \in H_{c,o}(U \setminus A, \mu_o|_{U \setminus A})$;

(3) $f \in \mathrm{Ker}\, \theta_{U \setminus A}$;

where $A = \phi(B_o(1) \cap E^n)$ and $2A = \phi(B_o(2) \cap E^n)$.

Using the differentiable structure that $\phi : E^n \hookrightarrow M^n$ imposes on $\phi(E^n)$, we can construct an open subset N of M such that

(4) $\mathrm{Cl}\, N \subset U$;

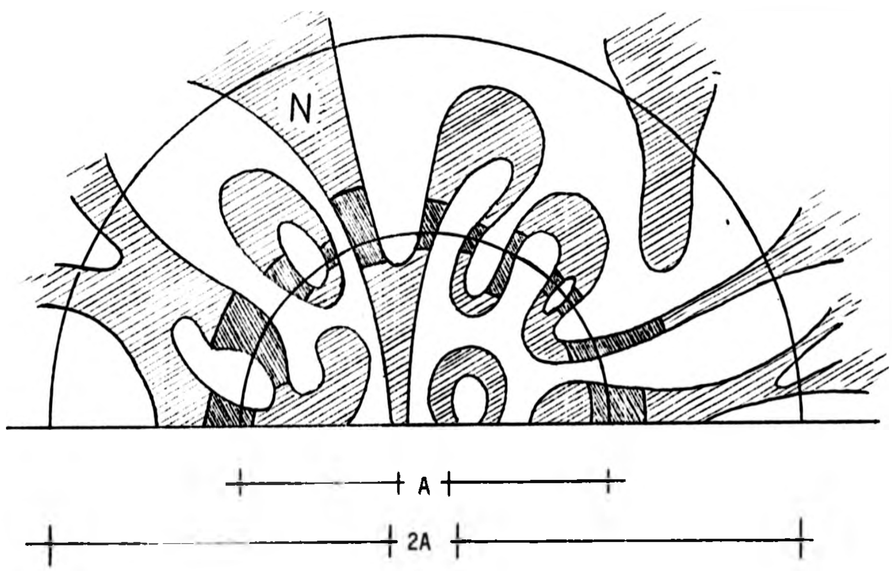
(5) $h \in H_{c,o}(N, \mu_o|_N)$;

(6) $h \in \mathrm{Ker}\, \theta_N$;

(7) $\mathrm{Cl}(N \cap \mathrm{Fr}\, A)$ is an $(n-1)$-dimensional manifold such that there is an embedding

(a) $\mathrm{Cl}(N \cap \mathrm{Fr}A) \times [-1, 2] \hookrightarrow \mathrm{Cl}\, N \cap 2A$, satisfying the following property:

(b) $(N \cap \mathrm{Fr}A) \times [-1, 2]$ is mapped into N in such a way that $(N \cap \mathrm{Fr}A) \times \langle -1, 2 \rangle \hookrightarrow N$ is a bicollar of $(N \cap \mathrm{Fr}A) \times \{1/2\} = N \cap \mathrm{Fr}A$ in N.



In other words, we have engineered U into N in order to canalize the flow of mass in and out $\mathrm{Fr}A$ through channels with reasonable embankments.

Assume, without loss of generality, that

$\mu_o|_{\mathrm{Cl}(N \cap \mathrm{Fr}A) \times [-1,2]} = \nu \times dt$, where ν is a measure in

$Cl(N \cap Fr\ A)$ and dt is Lebesgue measure on $[-1,2]$.

By corollary 8.6, $\text{Ker}\ \theta_N$ is connected and hence is generated, as a group, by any neighbourhood of the identity. Expressing h as a product of small homeomorphisms and replacing it by one of the factors, we can suppose further that

$$h((N \cap Fr\ A) \times [1/4,\ 3/4]) \subset (N \cap Fr\ A) \times \langle o,1\rangle \quad ,$$

Define

$$X_1 = (N \cap A) \cup (N \cap Fr\ A) \times [1/2,\ 1\rangle \quad ;$$

$$X_2 = (N \setminus A) \cup (N \cap Fr A) \times \langle o,1/2] \quad ;$$

$$L = X_1 \cap X_2 = (N \cap Fr\ A) \times \langle o,\ 1\rangle \quad .$$

Applying Lemma 8.11 to each component of $Cl(N \cap FrA)$ and using a small isotopy of h in $\text{Ker}\ \theta_N$, we can find a homeomorphism b of M in $\mathcal{H}_{c,o}(L,\ \mu_o|_L)$ such that b agrees with h on $(N \cap Fr\ A) \times [1/4,\ 3/4]$.

Since the composite $b^{-1} \circ h$ is the identity on $(N \cap Fr\ A) \times [1/4,\ 3/4]$, we can find two homeomorphisms, say g'' and f', such that

$$(8) \quad b^{-1} \circ h = g'' \circ f' \quad ;$$

$$(9) \quad g'' \in \mathcal{H}_{c,o}(X_1,\ \mu_o|_{X_1}) \quad ;$$

$$(10) \quad f' \in \mathcal{H}_{c,o}(X_2,\ \mu_o|_{X_2}) \quad .$$

Let $g' = b \circ g''$ and observe that although h is equal to the product $g' \circ f'$ and conditions (1) and (2) above are satisfied, it may well happen that $\theta_{X_2}(f')$ is not zero.

To remedy this situation, we want to perturb f' and g'. For this purpose we consider the following commutative diagram

$$\begin{array}{ccccc}
\tilde{H}^{\partial}_{c,o}(L,\mu_o|_L) & \xrightarrow{\Psi_1} & \tilde{H}_{c,o}(X_1,\mu_o|_{X_1}) \times \tilde{H}_{c,o}(X_2,\mu_o|_{X_2}) & \xrightarrow{\Phi_1} & \tilde{H}_{c,o}(N,\mu_o|_N) \\
\Big\downarrow \tilde{\theta}^{\partial}_L & & \Big\downarrow \tilde{\theta}_{X_1} \times \tilde{\theta}_{X_2} & & \Big\downarrow \tilde{\theta}_N \\
H_1(L, \mathbb{R}) & \xrightarrow{\Psi_2} & H_1(X_1, \mathbb{R}) \oplus H_1(X_2, \mathbb{R}) & \xrightarrow{\Phi_2} & H_1(N, \mathbb{R})
\end{array}$$

where

$$\Psi_1(P) = (P, P^{-1})$$

$$\Phi_1(G, F) = G \circ F$$

$$\Psi_2(z) = (z, -z)$$

$$\Phi_2(v, w) = v + w$$

The Mayer-Vietoris theorem implies that Image Ψ_2 = Kernel Φ_2.

Suppose now that $G \in \tilde{H}_{c,o}(X_1,\mu_o|_{X_1})$ and $F \in \tilde{H}_{c,o}(X_2,\mu_o|_{X_2})$ are given such that $\tilde{\theta}_N(G \circ F) = o$. Then, by the exactness of the Mayer Vietoris sequence and the fact that $\tilde{\theta}^{\partial}_L$ is surjective (see 8.4 and 8.7) we can find a $P \in \tilde{H}^{\partial}_{c,o}(L,\mu_o|_L)$ with

$\Psi_2(\tilde{\theta}_L^{\partial}(P)) = (\tilde{\theta}_{X_1}(G), \tilde{\theta}_{X_2}(F))$. Therefore $P \circ F \in \operatorname{Ker} \tilde{\theta}_{X_2}$ and $(G \circ P^{-1}) \circ (P \circ F) = G \circ F$.

This concludes the proof.

□

8.12 Definition.

Let $\mathcal{U}$ be an open covering of a manifold M. A homeomorphism of M is said to be $\mathcal{U}$-small if its support is contained in some element of $\mathcal{U}$.

8.13 Remarks

Let $\mathcal{U}$ be an open covering of a manifold M. By using balls of small diameter in the proof of 8.10, we can add in Proposition 8.10 that each h_i in the decomposition of h is $\mathcal{U}$-small.

Collecting together the results obtained in 8.4 and 8.10 we can state the following theorem.

8.14 Theorem.

Let M^n be a second countable manifold and let $\mu_o \in M_g^{\partial}(M)$. The map

$$\theta : \mathcal{H}_{c,o}(M, \mu_o) \to H_1(M, \mathbb{R})/\Gamma_M$$

is surjective. The kernel of θ is generated as a group by its elements having support in n-cells.

Moreover, given any open covering $\mathcal{U}$ of M, we can write each element of Ker θ as a composition of μ_o-preserving homeomorphisms which are $\mathcal{U}$-small.

□

8.15 Remark.

If M has non-empty boundary, then the same result holds for

$$\theta^\partial : H^\partial_{c,o}(M, \mu_o) \to H_1(M, \mathbb{R})$$

8.16 Theorem.

Let M^n be a connected, second countable manifold without boundary and of dimension $n \geq 3$. Let $\mu_o \in M^\partial_g(M)$. Then the kernel of the map

$$\theta : H_{c,o}(M, \mu_o) \to H_1(M, \mathbb{R})/\Gamma_M$$

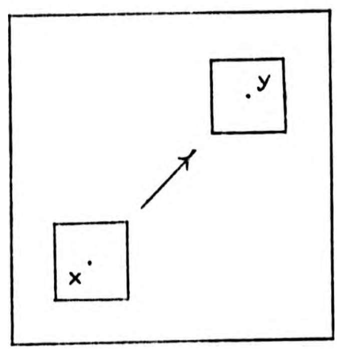
is a simple group.

Furthermore, it is the smallest non-trivial normal subgroup of $H_{c,o}(M, \mu_o)$ and it is also equal to the commutator subgroup of $H_{c,o}(M, \mu_o)$.

This theorem is an immediate consequence of the following three results:

(A) Let x and y be two points in M. Then there exists a homeomorphism $h \in H_{c,o}(M, \mu_o)$ which sends x into y and is supported in a topological n-ball.

Remarks. This result (which holds in all dimensions greater than one) was essentially proved by Oxtoby an Ulam in [20,p.895] . It also follows from what it has been said here: Let B be a ball in M such that $\{x,y\} \subset \mathring{B}$. Assume, without loss of generality that $B = I^n$ and $\mu_o|_B$ is a multiple of Lebesgue measure ,



By sliding a small cube centred at x onto a small cube centred at y and then applying the extension of isotopies theorem 5.5 we get the desired homeomorphism.

In particular, (A) proves that Ker θ acts transitively on M (see 8.9).

(B) Let ℓ be Lebesgue measure on the interior of the unit cube I^n . Then the group $H_c(\mathring{I}^n, \ell)$ is perfect (i.e. it is equal to its commutator subgroup).

Remarks. This non-trivial result is Theorem 7.4 in Fathi [9 , p. 88]. In his paper, A. Fathi, poses as an open question whether or not (B) holds in dimension two. His question is stated in a more precise manner.

Using (B) and the von Neumann-Oxtoby-Ulam Theorem, it follows that Ker θ is perfect (see 8.14).

(C) Let K, N be two subgroups of the full group of homeomorphisms $\mathcal{H}(M)$ satisfying the following properties:

(1) N is not trivial;

(2) K acts on N by conjugation (i.e. K is contained in the normalizer of N in $\mathcal{H}(M)$);

(3) K acts transitively on M;

(4) For any open covering $\mathcal{U}$ of M, K is generated by its $\mathcal{U}$-small elements.

Then, the commutator subgroup of K is contained in N.

In particular, if K is perfect, then K is simple.

Remark. This is just a restatement of a well known argument due to D.B.A. Epstein.

Proof. (c.f. Fathi [9 , p.90]).

Let f_0 be a non-trivial element in N. Then we can find a non-empty open subset $V_0 \subset M$ such that

(5) $V_0 \cap f_0(V_0) = \emptyset$.

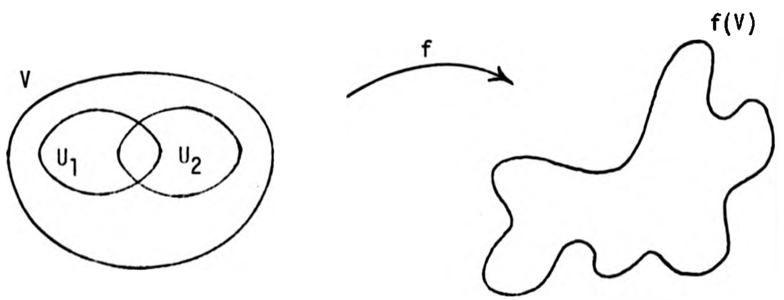
Let $\mathcal{V} = \{b(V_0) \mid b \in K\}$. By (3), $\mathcal{V}$ is a covering of M.

Let $\mathcal{U}$ be an open barycentric refinement of $\mathcal{V}$. Hence, if $U_1, U_2 \in \mathcal{U}$ and $U_1 \cap U_2 \neq \emptyset$, then $U_1 \cup U_2 \subset V$ for some $V \in \mathcal{V}$.

By (4), it is enough to show that if $h, g \in K$ are $\mathcal{U}$-small, then the commutator $[h,g] = h\, g\, h^{-1} g^{-1}$ belongs to N.

Let $U_1, U_2 \in \mathcal{U}$ be such that supp $h \subset U_1$ and supp $g \subset U_2$. If $U_1 \cap U_2 = \emptyset$, then $[h,g] = \mathrm{Id}$ and we are done. Otherwise, let $b \in K$ with $U_1 \cup U_2 \subset b(V_0) = V$.

By (2) and (5), we have that if we put $f = b\, f_0 b^{-1}$ then $f \in N$ and $V \cap f(V) = \emptyset$.



Using (2) we see that $[h, f] \in N$, because

$$[h, f] = h\, f\, h^{-1} f^{-1} = (hfh^{-1})\, f^{-1} .$$

By (2) again, the commutator $[[h,f] , g]$ lies in N also.

Since $\operatorname{supp} f h^{-1} f^{-1} \subset f(V)$, it follows that $fh^{-1} f^{-1}$ commutes with g.

Finally,

$$\begin{aligned} [[h, f], g] &= h(fh^{-1} f^{-1})\, g(fh\, f^{-1})h^{-1} g^{-1} \\ &= h\, g(fh^{-1} f^{-1})(fh\, f^{-1})\, h^{-1} g^{-1} \\ &= h\, g\, h^{-1} g^{-1} \\ &= [h, g] \end{aligned}$$

proving that $[h, g] \in N$.

$\square$

Off-print. Measures on σ-compact manifolds and their equivalence under homeomorphisms.

Appendix One.

It is the purpose of this appendix to describe a particular class of direct limit spaces which includes all those which are considered in the present work (see Whitehead [25]).

Let X be a topological space and $\{ A_\alpha \mid \alpha \in J \}$ a family of closed subsets whose union is X. We say that X has the strong topology with respect to the A_α if and only if it satisfies the following condition: a subset B of X, whose intersection with each of the sets A is closed, is itself closed.

Now let X be a set, and let $\{ A_\alpha \mid \alpha \in J \}$ be a collection of topological spaces, each a subset of X. We say that $\{ A_\alpha \}$ is a coherent family (of topological spaces) on X if and only if

(a) $X = \cup_\alpha A_\alpha$;

(b) $A_\alpha \cap A_\beta$ is a closed subset of A_α, for every $\alpha, \beta \in J$;

(c) For every $\alpha, \beta \in J$, the topologies of A_α and A_β agree on $A_\alpha \cap A_\beta$.

The topology coinduced on X by the family $\{ A_\alpha \}$ is the largest or finest topology such that, for each $\alpha \in J$, the inclusion

$$i_\alpha : A_\alpha \hookrightarrow X ,$$

$$i_\alpha (x) = x \quad \forall\ x \in A_\alpha .$$

is continuous. With this topology, X satisfies the following properties:

(1) $i_\alpha : A_\alpha \hookrightarrow X$ is a closed embedding, for each $\alpha \in J$;

(2) Every closed subspace B in X has the strong topology with respect to the $A_\alpha \cap B$;

(3) If Y is a space and $X \xrightarrow{f} Y$ a function, then f is continuous if and only if the composition (resp. restriction) $f \circ i_\alpha : A_\alpha \to Y$ (resp. $f|_{A_\alpha} : A_\alpha \to Y$) is continuous for each $\alpha \in J$. In particular, suppose $\{ B_\beta \}_{\beta \in L}$ is a coherent family of spaces on the set Y and $f: X \to Y$ is <u>stratified</u> in the sense that for each $\alpha \in J$ $f(A_\alpha) \subset B_\beta$ for some $\beta \in L$ and $f|_{A_\alpha} : A_\alpha \to B_\beta$ is continuous. Then f is continuous when Y is given the coinduced topology by the family $\{ B_\beta \}$;

(4) If $J' \subset J$ is such that for each $\alpha \in J$ there exists a finite sequence in J', say $\beta_1, \beta_2, \ldots, \beta_n$ with $A_\alpha \subset \bigcup_{i=1}^{n} A_{\beta_i}$ then the topology coinduced on X by the (coherent) family $\{ A_\beta \mid \beta \in J' \}$ agrees with the one coinduced on X by $\{ A_\alpha \mid \alpha \in J \}$;

(5) If J is countable and X is Hausforff then every compact set K in X is contained in some finite union $\bigcup_{i=1}^{n} A_{\alpha_i}$ $\alpha_i \in J$.

<u>Remarks on subspaces and products.</u>

Let $\{ A_\alpha \}_{\alpha \in J}$ be a coherent family of spaces on the set X and let B be a subset of X. Then $\{ A_\alpha \cap B \}_{\alpha \in J}$ is a coherent family of spaces on B and the inclusion $B \hookrightarrow X$ is continuous, but not an embedding in general when B(resp. X) is given the coinduced topology by the family $\{ A_\alpha \cap B \}_{\alpha \in J}$ (resp. $\{ A_\alpha \}_{\alpha \in J}$). (Compare with (1) and (2) above).

In this work, a subset $B \subset X$ is always given the topology coinduced by the $A_\alpha \cap B$ (and not the topology it inherits as a subspace of X).

Let $\{ A_\alpha \}_{\alpha \in J}$, $\{ B_\beta \}_{\beta \in L}$ be coherent families of spaces on the sets X and Y respectively. Then the product $X \times Y$ can be topologized in two generally distinct ways.

First, we can topologize X and Y via the families $\{ A_\alpha \}$ and $\{ B_\beta \}$ and then give $X \times Y$ the standard cartesian product topology.

Secondly, we can give $X \times Y$ the topology coinduced by the coherent family $\{ A_\alpha \times B_\beta \mid (\alpha, \beta) \in J \times L \}$.

In this work we always choose the second procedure.

<u>Note</u>:

If we think the pairs $(X, \{ A_\alpha \}_{\alpha \in J})$ where X is a set and $\{ A_\alpha \}$ a coherent family of spaces on X as a category having for morphisms the class of stratified functions, then we have just described a (covariant) functor from this category into the category of topological spaces and continuous maps that does not commute with (categorical) products in general.

Appendix Two.

Compactly generated spaces are examples of the "stratified spaces" considered in the previous appendix. Here we list some elementary properties of those topologies which are determined by its compact subsets.

Let X be a topological Hausdorff space. We say that X is a k-space (or compactly generated space) if X has the strong topology with respect to its family of compact subsets. First countable spaces and locally compact spaces are examples of k-spaces.

If X is a Hausdorff space, the associated compactly generated space is the space $k(X)$ defined as follows: $k(X)$ and X have the same underlying set, and $k(X)$ has the topology coinduced by the (coherent) family of compact subsets of X. If Y is Hausdorff and $f : X \to Y$ any function, let $k(f)$ be the same function regarded as a map of $k(X)$ into $k(Y)$.

The following simple properties are satisfied:

(1) The identity map $k(X) \to X$ is continuous;

(2) $k(X)$ is compactly generated;

(3) If X is compactly generated, then $k(X) = X$;

(4) $k(X)$ and X have the same compact sets;

(5) If $f : X \to Y$ is a function, then $k(f)$ is continuous if and only if $f|_K : K \to Y$ is continuous for every compact set $K \subset X$;

(6) A Hausdorff space Y is a k-space if and only if it is the quotient of a locally compact Hausdorff space, (see Dugundji [6, Chapter XI, pp. 247-249]);

(7) If Y is a locally compact space, then $k(X \times Y) = k(X) \times Y$. In particular, if X is a k-space (and Y locally compact), then $X \times Y$ is a k-space;

(8) If X is a k-space, then all open and all closed subsets are compactly generated.

Note.

In Whitehead [25 , 4.15, p.20] it is only asserted that regular open subsets of k-spaces are k-spaces, and although this is enough for our applications, the proof for an arbitrary open subset is simple. Indeed, let $U \subset X$ be open and let $\rho: Z \twoheadrightarrow X$ be an identification map where Z is locally compact. Then $\rho|\rho^{-1}(U): \rho^{-1}(U) \twoheadrightarrow U$ is an identification map (see Dugundji [6, VI(2.1), p.122]), hence by (6), U is a k-space for $\rho^{-1}(U)$ is certainly locally compact);

(9) Let $U \subset X$ be an open subspace. Then kU is an open subspace of kX.

<u>Proof:</u>

By (5) above, $kU \xhookrightarrow{ki} kX$ is continuous. Since U is open in X then, by (1), it is open in kX. By (8), U is a k-space in kX. Hence $kU \hookrightarrow kX$ is an embedding.

□

(10) Let $\{B_\beta\}_{\beta \in \Gamma}$ be a coherent family of k-spaces in the set Y. If Y is Hausdorff when given the topology coinduced by the B_β, then Y is also compactly generated. Furthermore, Y has the strong topology with respect to the family

$$\{K \subset Y \mid K \text{ is a compact subset of } B_\beta \text{ for some } \beta \in \Gamma\}.$$

If U is an open subspace of Y, then U has the strong topology with respect to the family

$$\{B_\beta \cap U \mid \beta \in \Gamma\}.$$

This follows from (8) above.

If Γ is countable, then every compact subset $K \subset Y'$ is contained in some finite union $\bigcup_{i=1}^{n} B_{\beta_i}$.

(11) Let Y_τ be a Hausdorff space, and let $\{B_\beta\}_{\beta \in \Gamma}$ be a closed covering of Y_τ consisting of k-spaces. Suppose that any compact subset $K \subset Y_\tau$ is contained in some finite union $\bigcup_{i=1}^{n} B_{\beta_i}$. Then

$$Y_{\underrightarrow{\lim}} = k(Y_\tau)$$

where $\underrightarrow{\lim}$ denotes the topology on Y coinduced by the B_β.

This follows from (10) and the fact that, by hypothesis, the inclusion $Y_{\underrightarrow{\lim}} \hookrightarrow Y_\tau$ is a proper map.

Index of Notation.

Let X be a locally compact, second countable, Hausdorff space (see 1.2, 1.6, 1.14, 3.1, and 3.7).

$M(X)$	set of Radon measures on X ;
$M_g(X)$	set of good measures on X

Let $\mu_o \in M_g(X)$ and let $K \subset X$ be compact.

$M_g(K, X, \mu_o)$	set of good measures μ on X which are K-related to μ_o.
$M_{g,c}(X, \mu_o)$	set of good measures μ on X which are compactly related to μ_o;
$M_g(X, \alpha(\mu_o))$	set of good measures μ on X such that $\alpha(\mu)=\alpha(\mu_o)$ and $\mu(X_j)=\mu_o(X_j)$ for each connected component X_j of X.
$M(X, <<\lambda)$	set of Radon measures on X absolutely continuous with respect to the Radon measure λ on Y.
$M(X, <<\lambda<<)$	set of Radon measures μ on X which have the same sets of measure zero as λ .
$M_g(X, \mu_o\text{-e-reg})$	$M_g(X, \alpha(\mu_o)) \cap M(X, <<\mu_o<<)$
$M_g(K, X, \mu_o\text{-e-reg})$	$M_g(K, X, \mu_o) \cap M(X, <<\mu_o<<)$
$M_{g,c}(X, \mu_o\text{-e-reg})$	$M_{g,c}(X, \mu_o) \cap M(X, <<\mu_o<<)$

Let M be a second countable manifold

$\mathcal{M}^{\partial}(M)$	set of Radon measures μ on M such that $\mu(\partial M) = o$
$\mathcal{M}_g^{\partial}(M)$	$\mathcal{M}^{\partial}(M) \cap \mathcal{M}_g(M)$

Let $\mu_o \in \mathcal{M}_g^{\partial}(M)$ and let $K \subset M$, be compact.

$\mathcal{M}_g^{\partial}(K, M, \mu_o)$	$\mathcal{M}^{\partial}(M) \cap \mathcal{M}_g(K, M, \mu_o)$;
$\mathcal{M}_{g,c}^{\partial}(M, \mu_o)$	$\mathcal{M}^{\partial}(M) \cap \mathcal{M}_{g,c}(M, \mu_o)$;
$\mathcal{M}_g^{\partial}(M, \alpha(\mu_o))$	$\mathcal{M}^{\partial}(M) \cap \mathcal{M}_g(M, \alpha(\mu_o))$;
$\mathcal{M}_g^{\partial}(M, \mu_o\text{-e-reg})$	$\mathcal{M}_g\ (M, \mu_o\text{-e-reg})$;
$\mathcal{M}_g^{\partial}(K, M\ \mu_o\text{-e-reg})$	$\mathcal{M}_g(K, M, \mu_o\text{-e-reg})$;
$\mathcal{M}_{g,c}^{\partial}(M, \mu_o\text{-e-reg})$	$\mathcal{M}_{g,c}(M, \mu_o\text{-e-reg})$.

Let X be a locally compact, locally connected, second countable, Hausdorff space (see 1.1, 1.13, 3.1, 3.9).

$\mathcal{H}(X)$	group of homeomorphisms of X.

Let $\mu_o \in \mathcal{M}_g(X)$ and let $K \subset X$ be compact.

$\mathcal{H}(K, X)$	group of homeomorphisms of X supported in K.
$\mathcal{H}_c(X)$	group of compactly supported homeomorphisms of X.
$\mathcal{H}(X, \mu_o\text{-e-reg})$	group of homeomorphisms h of X such that $h_* \mu_o \in \mathcal{M}_g(X, \mu_o\text{-e-reg})$;

$H(K, X, \mu_o\text{-e-reg})$	$H(K, X) \cap H(X, \mu_o\text{-e-reg})$;
$H_c(X, \mu_o\text{-e-reg})$	$H_c(X) \cap H(X, \mu_o\text{-e-reg})$;
$H(X,\mu_o)$	group of homeomorphisms of X preserving μ_o;
$H(K, X,\mu_o)$	$H(K, X) \cap H(X, \mu_o)$;
$H_c(X, \mu_o)$	$H_c(X) \cap H(X, \mu_o)$;
$H_o(X)$	path connected component of the identity in $H(X)_\kappa$ where κ is the compact-open topology (i.e. group of homeomorphisms of X isotopic to the identity).
$H_{c,o}(X)$	path connected component of the identity in $H_c(X)_{\underrightarrow{\lim}}$; where $\underrightarrow{\lim}$ is the limit topology (i.e. group of homeomorphisms of X compactly supported and compactly isotopic to the identity);
$H_o(X, \mu_o)$	path connected component of the identity in $H(X,\mu_o)_\kappa$, where κ is the compact open topology.
$H_{c,o}(X, \mu_o)$	path connected component of the identity in $H_c(X,\mu_o)_{\underrightarrow{\lim}}$, where $\underrightarrow{\lim}$ is the direct limit topology.

Let M be a second countable manifold, $K \subset M$ be compact, and let $\mu_o \in M_g^\partial(M)$.

$H^\partial(M)$	group of homeomorphisms of X fixing ∂M pointwise;

$H^{\partial}(K, M)$	$H^{\partial}(M) \cap H(K, M)$;
$H_c^{\partial}(M)$	$H^{\partial}(M) \cap H_c(M)$;
$H^{\partial}(M, \mu_o\text{-e-reg})$	$H^{\partial}(M) \cap H(M, \mu_o\text{-e-reg})$;
$H^{\partial}(K, M, \mu_o\text{-e-reg})$	$H^{\partial}(M) \cap H(K, M, \mu_o\text{-e-reg})$;
$H_c^{\partial}(M, \mu_o\text{-e-reg})$	$H^{\partial}(M) \cap H_c(M, \mu_o\text{-e-reg})$;
$H^{\partial}(M, \mu_o)$	$H^{\partial}(M) \cap H(M, \mu_o)$;
$H^{\partial}(K, M, \mu_o)$	$H^{\partial}(M) \cap H(K, M, \mu_o)$
$H_c^{\partial}(M, \mu_o)$	$H^{\partial}(M) \cap H_c(M, \mu_o)$;
$H_o^{\partial}(M)$	path connected component of the identity in $H^{\partial}(M)_\kappa$, where κ is the compact open topology.
$H_{c,o}^{\partial}(M)$	path connected component of the identity in $H_c^{\partial}(M)_{\underrightarrow{\lim}}$ where $\underrightarrow{\lim}$ is the direct limit topology.
$H_o^{\partial}(M, \mu_o)$	path connected component of the identity in $H(M,\mu_o)_\kappa$ where κ is the compact open topology.
$H_{c,o}^{\partial}(M, \mu_o)$	path connected component of the identity in $H_c^{\partial}(M, \mu_o)_{\underrightarrow{\lim}}$, where $\underrightarrow{\lim}$ is the direct limit topology.

Let M be a second countable manifold, and let A, B be subsets of M. Let $\mu_o \in M_g^{\partial}(M)$ (see 4.6).

$I(A, M)$	space of embeddings $\iota: A \hookrightarrow M$ such that $\iota^{-1}(\partial M) = A \cap \partial M$, endowed with the compact open topology.

$I(A, M; \ll\mu_o\ll)$	subspace of embeddings ι in $I(A, M)$ such that $\iota^*\mu_o$ has the same subsets of measure zero as $\mu_o\|_A$;
$I(A, B, M)$	subspace of embeddings ι in $I(A, M)$ such that ι is the identity in $A \cap B$;
$I(A, B, M; \ll\mu_o\ll)$	$I(A, B, M) \cap I(A, M, \ll\mu_o\ll)$.
$I(A, M; \mu_o)$	space of embeddings ι in $I(A, M)$ such that $\iota^*\mu_o = \mu_o\|_A$.

Let (X, x_o) be a topological pointed space (see 6.1).

$P(X, x_o) = P(X)$	space of paths of X based at x_o .
$\Omega(X, x_o) = \Omega(X)$	space of loops of X based at x_o .
$\widetilde{(X, x_o)} = \tilde{X}$	space of equivalence classes of paths in $P(X)$ under homotopy relative to $\{o,1\}$.
$\Pi(X, x_o) = \Pi(X)$	fundamental group of X at x_o .
$IS(X)$	space of isotopies of X .
$IS_c(X)$	space of compactly supported isotopies of X .

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MEASURES ON SIGMA-COMPACT MANIFOLDS AND THEIR EQUIVALENCE UNDER HOMEOMORPHISMS

R. BERLANGA AND D. B. A. EPSTEIN

The von Neumann–Oxtoby–Ulam theorem states that the group of homeomorphisms of a (topological) compact manifold M acts transitively on the set of "∂-good" probability measures on M. We generalize this result to the σ-compact case.

When M is σ-compact the action considered is also transitive on the set of probability measures, but not on the set of infinite measures in general. An example of this phenomenon is the fact that there is no homeomorphism of $(-\infty, \infty)$ onto $(0, \infty)$ preserving Lebesgue measure.

The problem of giving a simple characterization of those measures in the unit n-dimensional cube I^n, equivalent to Lebesgue measure under a homeomorphism of I^n, was originally proposed by Oxtoby and Ulam in 1936. Their initial conjecture was proved at that time by J. von Neumann, but was never published. In 1941, Oxtoby and Ulam published, for the first time, a proof based on different considerations and extended the result to regularly connected compact polyhedra and euclidean spaces (see Oxtoby and Ulam [6]). Their work was further extended by Fathi [3], whose paper inspired ours.

Introduction

Let M^n be an n-dimensional manifold, possibly with non-empty boundary ∂M. A *Radon measure* μ on M is a locally finite positive measure defined on the σ-algebra of all Borel subsets. The *support* of μ is the complement of the maximum open subset of M in which μ has zero measure. We say that μ is a *good measure* if it has no atoms (that is, points of non-zero μ-measure) and if its support is the whole of M. A locally finite measure on a σ-compact manifold is automatically regular (see Rudin [7]).

Denote by $\mathscr{H}(M)$ (respectively $\mathscr{H}^{\partial}(M)$) the group of homeomorphisms of M (respectively the group of homeomorphisms of M fixing ∂M point-wise) and by $\mathscr{M}(M)$ (respectively $\mathscr{M}_g(M)$) the set of all Radon measures on M (respectively the set of good Radon measures on M). Also let $\mathscr{M}^{\partial}(M) = \{\mu \in \mathscr{M}(M) \mid \mu(\partial M) = 0\}$ and $\mathscr{M}_g^{\partial}(M) = \mathscr{M}^{\partial}(M) \cap \mathscr{M}_g(M)$.

There is an action

$$\mathscr{H}(M) \times \mathscr{M}(M) \longrightarrow \mathscr{M}(M)$$

$$(h, \mu) \longmapsto h_* \mu$$

where $h_* \mu$ is defined by $h_* \mu(E) = \mu(h^{-1}(E))$ for each Borel set $E \subset M$.

Before we can state our main result, we must recall the concept of an "end of a (σ-compact) connected manifold". An *end of M* is a function e which assigns to each compact subset K of M a non-empty connected component $e(K)$ of $M \setminus K$, in such a way that $K_1 \subset K_2$ implies that $e(K_2) \subset e(K_1)$. Let $\mathscr{E}(M)$ be the set of all ends and define a topology on $M \cup \mathscr{E}(M)$ by defining, for each compact K, a typical

Received 11 September, 1981.

J. London Math. Soc. (2), 27 (1983), 63–74

neighbourhood $N_K(e_0)$ of an end e_0 as the set $e_0(K) \cup \{\text{ends } e \mid e(K) = e_0(K)\}$. With this topology $M \cup \mathscr{E}(M)$ is a compact Hausdorff space containing $\mathscr{E}(M)$ as a closed subset.

Let $h \in \mathscr{H}(M)$. Then h induces a homeomorphism $h_0 : \mathscr{E}(M) \to \mathscr{E}(M)$ such that $h_0(e)(K) = h(e(h^{-1}(K)))$ for all $e \in \mathscr{E}(M)$ and for all compact $K \subset M$. Clearly the function $h \mapsto h_0$ is a group homomorphism from $\mathscr{H}(M)$ into the group $\mathscr{H}(\mathscr{E}(M))$ of homeomorphisms of $\mathscr{E}(M)$.

Now let $\mu \in \mathscr{M}(M)$ and define $\alpha(\mu) : \mathscr{E}(M) \to \{1, \infty\}$ such that

$$\alpha(\mu)(e) = \begin{cases} \infty & \text{if } \mu(e(K)) = \infty \text{ for all compact } K, \\ 1 & \text{if } \mu(e(K)) < \infty \text{ for some compact } K \end{cases}$$

(observe that $\alpha(\mu)^{-1}(1)$ is open in $\mathscr{E}(M)$).

It is easy to verify that if $h \in \mathscr{H}(M)$ and $\mu \in \mathscr{M}(M)$ then $\alpha(\mu) \circ h_0^{-1} = \alpha(h_* \mu)$. Hence we have a commutative diagram

$$\begin{array}{ccc} \mathscr{H}(M) \times \mathscr{M}(M) & \longrightarrow & \mathscr{M}(M) \\ \downarrow & & \downarrow \\ \mathscr{H}(\mathscr{E}(M)) \times F(\mathscr{E}(M), \{1, \infty\}) & \longrightarrow & F(\mathscr{E}(M), \{1, \infty\}) \end{array}$$

where $F(\mathscr{E}(M), \{1, \infty\})$ denotes the set of functions from $\mathscr{E}(M)$ into $\{1, \infty\}$.

The main theorem

THEOREM. *Let M be a connected, σ-compact manifold and let $\mu_1, \mu_2 \in \mathscr{M}_g^\partial(M)$ be two (∂-good) measures with the same total mass and such that they agree on ends (that is, $\alpha(\mu_1) = \alpha(\mu_2)$). Then there exists a homeomorphism $h \in \mathscr{H}^\partial(M)$ fixing ends and isotopic to the identity, such that $h_* \mu_1 = \mu_2$.*

COROLLARY. *Let $\mu_1, \mu_2 \in \mathscr{M}_g^\partial(M)$. Then there exists $k \in \mathscr{H}^\partial(M)$ such that $k_* \mu_1 = \mu_2$ if and only if*

(1) *$\mu_1(M) = \mu_2(M)$, and*

(2) *$\alpha(\mu_1) \circ h_0^{-1} = \alpha(\mu_2)$ for some $h \in \mathscr{H}^\partial(M)$.*

Lemmas and propositions

We want to prove first that the von Neumann–Oxtoby–Ulam theorem holds for a slightly more general class of compact spaces than the class of compact connected manifolds. For this purpose we introduce the following notation when $X \subset M$.

We define $\mathring{X}$ to be the topological interior of X in M, Int X to be $(M \setminus \partial M) \cap \mathring{X}$, Cl X to be the closure of X in M, Fr X to be the topological frontier of X in M, and BX to be $(\partial M \cap X) \cup \text{Fr } X$. Whenever X is closed, we define $\mathscr{M}_g^B(X)$ to be the set of all measures $\mu \in \mathscr{M}(M)$ such that the support of μ is X, μ has no atoms and $\mu(BX) = 0$, and define $\mathscr{H}^B(X)$ to be the set of homeomorphisms on X which fix BX

pointwise. Note that $\mathscr{H}^B(X)$ can be identified with the set $\{h \in \mathscr{H}^c(M) \mid \operatorname{supp} h \subset X\}$ where $\operatorname{supp} h = \mathrm{Cl}\{x \in M \mid f(x) \neq x\}$.

DEFINITIONS. Call any subset $K \subset M$ a (*closed*) *n-cell* if it is homeomorphic to the unit n-cube $I^n = [0, 1]^n$.

A set $K \subset M^n$ is called a *relative n-cell* if there exists a continuous function $\phi : I^n \to K$ such that

(1) ϕ is onto,

(2) ϕ restricted to $\operatorname{Int} I^n$ is a homeomorphism onto its image,

(3) $\phi^{-1}\phi(\partial I^n) = \partial I^n$.

Note that $\mathrm{Cl}\,\phi(\operatorname{Int} I^n) = K$ and $\phi(\operatorname{Int} I^n) \cap \phi(\partial I^n) = \varnothing$; therefore $\phi(\partial I^n)$ has empty interior.

A set $K \subset M^n$ is called *measure conformable* if given $\mu_1, \mu_2 \in \mathscr{M}^B_\phi(K)$ of the same total mass, then there exists $h \in \mathscr{H}^B(K)$ such that $h_* \mu_1 = \mu_2$.

PROPOSITION 1. *Every relative n-cell K contained in a manifold M^n is measure conformable.*

To prove this proposition we need two lemmas due to Oxtoby and Ulam [5].

LEMMA 2 (proof taken from Fathi [3, p. 51]). *Suppose that μ is a measure on I^n such that $\mu(\partial I^n) = 0$, and suppose that S is a closed subset of I^n with empty interior. Then, for each δ and $\varepsilon > 0$, there exists a δ-homeomorphism h in $\mathscr{H}^c(I^n)$ such that $\mu(h(S)) < \varepsilon$.*

Proof. Using the fact that at most countably many elements in any disjoint collection of μ-measurable subsets of I^n can have positive μ-measure, we can find a subdivision $c_1, \ldots, c_k$ of I^n by cubes such that $\operatorname{diam}(c_i) < \delta$ and $\mu(\partial c_i) = 0$ $(i = 1, 2, \ldots, k)$. Let V be an open neighbourhood of $\bigcup_{i=1}^{k} \partial c_i$ such that $\mu(V) < \varepsilon$. Since S has empty interior, we can find, for each i, a small subcube $c_i' \subset \operatorname{Int} c_i$ with faces parallel to the faces of c_i and such that $S \cap c_i' = \varnothing$.

For each i, we can find a homeomorphism h_i of c_i satisfying

(1) $h_i | \partial c_i = \mathrm{Id}$,

(2) $h_i(c_i \setminus c_i') \subset V \cap c_i$.

Piecing together the h_i gives the desired homeomorphism h.

LEMMA 3 (cf. Fathi [3, Proposition 2.1, p. 51]). *Let K be a compact subset of M and let μ be a (Radon) measure on K with $\mu(B(K)) = 0$. If A is a closed set of M with $\mathring{A} \cap \mathring{K} = \varnothing$, then there exists a homeomorphism $h \in \mathscr{H}^c(M)$ such that*

$$\operatorname{supp} h \subset K, \qquad \mu(h(A \cap K)) = 0.$$

More precisely, the set

$$\{h \in \mathscr{H}^B(K) \mid \mu(h(A \cap K)) = 0\}$$

is a dense G_δ in $\mathscr{H}^B(K)$ in the compact open topology.

Proof. Let $\{B_i\}_{i \in \mathbb{N}}$ be a closed covering of $\operatorname{Int} K$ such that each B_i is a (closed) n-cell contained in $\operatorname{Int} K$ and such that $\mu(\operatorname{Fr} B_i) = 0$ for each $i \in \mathbb{N}$.

If $i \in \mathbb{N}$ and $j \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, define a subset $\mathscr{U}(i,j)$ of $\mathscr{H}^B(K)$ by

$$\mathscr{U}(i,j) = \{h \in \mathscr{H}^B(K) \mid \mu(h(A \cap K) \cap B_i) < 1/j\}.$$

We clearly have

$$\{h \in \mathscr{H}^B(K) \mid \mu(h(A \cap K)) = 0\} = \bigcap_{i,j} \mathscr{U}(i,j).$$

Thus we must show that $\mathscr{U}(i,j)$ is open and dense in $\mathscr{H}^B(K)$. Denseness of $\mathscr{U}(i,j)$ follows easily from Lemma 2. We prove now that $\mathscr{U}(i,j)$ is open.

Fix $h_0 \in \mathscr{U}(i,j)$. We can find an open neighbourhood V of $h_0(A \cap K) \cap B_i$ such that $\mu(V) < 1/j$. If h is close enough to h_0, we have $h(A \cap K) \cap B_i \subset V$, which implies that $\mu(h(A \cap K) \cap B_i) < 1/j$.

Proof of Proposition 1. Let $\phi : I^n \to K$ be a continuous surjection such that $\phi|\operatorname{Int} I^n$ is a homeomorphism onto $\phi(\operatorname{Int} I^n)$ and $\phi^{-1} \circ \phi(\partial I^n) = \partial I^n$. If $\nu \in \mathscr{M}_g^c(I^n)$ and X is a Borel subset of K, let $\phi_*(\nu)(X) = \nu(\phi^{-1}(X))$. Then $\phi_*(\nu) \in \mathscr{M}_g^B(K)$. In fact $\phi_* : \mathscr{M}_g^c(I^n) \to \mathscr{M}_g^B(K)$ is a one-one correspondence between $\mathscr{M}_g^c(I^n)$ and the set of measures μ in $\mathscr{M}_g^B(K)$ with $\mu(\phi(\partial I^n)) = 0$.

We have a well-defined (continuous) homomorphism $\tilde{\phi} : \mathscr{H}^c(I^n) \to \mathscr{H}^B(K)$ such that

$$\tilde{\phi}(h) = \begin{cases} \phi \circ h \circ \phi^{-1} & \text{on } \phi(\operatorname{Int} I^n) \\ \mathrm{Id} & \text{on } \phi(\partial I^n). \end{cases}$$

The diagram

$$\begin{tikzcd} \mathscr{H}^c(I^n) \times \mathscr{M}_g^c(I^n) \arrow[r] \arrow[d, "\tilde{\phi} \times \phi_*"] & \mathscr{M}_g^c(I^n) \arrow[d, "\phi_*"] \\ \mathscr{H}^B(K) \times \mathscr{M}_g^B(K) \arrow[r] & \mathscr{M}_g^B(K) \end{tikzcd}$$

clearly commutes.

Now let $\mu_1, \mu_2 \in \mathscr{M}_g^B(K)$. Since we can assume, without loss of generality by Lemma 3, that $(\mu_1 + \mu_2)(\phi(\partial I^n)) = 0$ the conclusion follows from the above discussion and the fact that I^n is measure conformable (see Oxtoby and Ulam [6]).

Remark. If $h : I \times I^n \to I^n$ is an isotopy such that $h_\tau \in \mathscr{H}^c(I^n)$ for all $\tau \in [0, 1]$, then $\tau \mapsto \tilde{\phi}(h_\tau)$ is an isotopy in K fixing BK pointwise. Since $\mathscr{H}^c(I^n)$ is contractible by Alexander's trick, it follows that for every $\mu_1, \mu_2 \in \mathscr{M}_g^B(K)$ there exists $h \in \mathscr{H}^B(K)$ isotopic to the identity such that $h_*(\mu_1) = \mu_2$.

Definitions. A subset X of an n-manifold M is *cellular* if for every neighbourhood U of X there is an n-cell Q such that $X \subset \operatorname{Int} Q \subset Q \subset U$. An

$(n-1)$-manifold B^{n-1} is *bi-collared* in M if there is a homeomorphism P of $B \times (-1, 1)$ onto a neighbourhood of B in M such that $P(b, 0) = b$, for all $b \in B$. If B is the boundary of a submanifold C^n of M, then $B \times (-1, 0]$ and $B \times [0, 1)$ denote the *inner* and *outer* collars of B. In general, we shall not distinguish $B \times \{t\}$ $(-1 < t < 1)$ from $P(B \times \{t\})$.

PROPOSITION 4. *Let M^n be a manifold with $n \geqslant 3$ and let $\{B_1, B_2, \ldots, B_k\}$ be a finite family of closed n-cells such that $\bigcup_{i=1}^{k} \operatorname{Int} B_i$ is connected. Then $B = \bigcup_{i=1}^{k} B_i$ is a relative n-cell (and therefore B is measure conformable).*

It would be interesting to know if Proposition 4 is true when $\dim M = 2$. The answer could have relevance to other problems.

In order to prove this proposition, we first prove two lemmas.

LEMMA 5 (Brown [2]). *Let C be a closed n-cell with bi-collared boundary ∂C in M^n $(n \geqslant 3)$ and let $\varepsilon, \delta > 0$. Suppose that $\mathscr{E} = \{E_1, \ldots, E_r\}$ is a finite family of sets in M such that each E_i is a closed n-cell of diameter less than $\varepsilon/2$ whose interior intersects C. Let $X = \{x_1, \ldots, x_t\}$ be a finite set of points in $\bigcup_{i=1}^{r} \operatorname{Int} E_i \setminus C$. Then there is a finite set of points $X' = \{x'_1, \ldots, x'_t\}$ in $\partial C \times (0, \delta)$ and an $\varepsilon/2$ homeomorphism $h : M \to M$ such that $\operatorname{supp} h \subset \bigcup_{i=1}^{r} \operatorname{Int} E_i \setminus C$ and $h(x'_i) = x_i$ $(i = 1, 2, \ldots, t)$.*

Proof. We may assume, without loss of generality, that the points $x_1, x_2, \ldots, x_t$ are distinct.

Associate with each x_i some element, say $E_{j(i)}$, of $\mathscr{E}$ which contains x_i in its interior. Associate with each E_j a point $y_j \in C \cap \operatorname{Int} E_j$. For $1 \leqslant i \leqslant t$ let α_i be a polygonal arc (relative to some combinatorial structure on $E_{j(i)}$) in $\operatorname{Int} E_{j(i)}$ from x_i to $y_{j(i)}$. Since an n-dimensional connected manifold cannot be disconnected by a subset of dimension less than or equal to $n-2$ (see Hurewicz and Wallman [4, Theorem IV.4, p. 48]), this can be done in such a manner that α_{i_1} and α_{i_2} are disjoint or intersect only in the common end point

$$y_{j(i_1)} = y_{j(i_2)}.$$

Let x'_i be a point of $\alpha_i \cap \partial C \times (0, \delta)$ such that the segment $[x_i, x'_i]$ of α_i does not intersect C. Since α_i is polygonal in $E_{j(i)}$, so is $[x_i, x'_i]$. Hence $[x_i, x'_i]$ is cellular in $E_{j(i)}$ and hence cellular in M. Hence there exist n-cells $Q_1, \ldots, Q_t$ such that

(1) $Q_i \cap Q_j = \varnothing$ if $i \neq j$, (2) $[x_i, x'_i] \subset \mathring{Q}_i$,

(3) $Q_i \cap C = \varnothing$, (4) $Q_i \subset \operatorname{Int} E_{j(i)}$.

Let h be a homeomorphism of M onto M such that

(1) $h\left| \left(M \setminus \bigcup_{i=1}^{t} Q_i\right) \right. = \mathrm{Id}$, (2) $h(Q_i) = Q_i$,

(3) $h(x'_i) = x_i$.

Then h is the required homeomorphism.

LEMMA 6 (Brown [2]). *Suppose that $0 < \gamma < 1$ and that the hypotheses of the above lemma are satisfied. Then there is an ε-homeomorphism f of M onto M such that $f(C) \supset f(\mathring{C}) \supset C \cup X$ and $\operatorname{supp} f \subset \bigcup_{i=1}^{r} \operatorname{Int} E_i \setminus C \cup \partial C \times (-\gamma, \gamma)$. In particular, f fixes pointwise the n-cell bounded by $\partial C \times \{-\gamma\}$.*

Proof. Choose δ such that $0 < 2\delta < \gamma$ and for each $c \in \partial C$ the diameter of $\{c\} \times [-2\delta, 2\delta]$—in the collar $C \times (-1, 1)$—is less than $\varepsilon/2$. Let g be a homeomorphism of M onto itself which is fixed on the cell bounded by $\partial C \times \{-2\delta\}$, stretches $\partial C \times \{0\}$ parametrically onto $\partial C \times \{\delta\}$ and is fixed outside $\partial C \times \{2\delta\}$. Then g is an $\varepsilon/2$-homeomorphism. Furthermore, if h is the homeomorphism obtained in the conclusion of the above lemma, then $f = h \circ g$ is the required ε-homeomorphism.

Proof of Proposition 4. Let $\mathscr{E}_1, \mathscr{E}_2, \ldots$ be a sequence of finite covers of $B = \bigcup_{l=1}^{k} B_l$ such that each element of $\mathscr{E}_i$ is a closed n-cell of diameter less than 2^{-i-1} and $\bigcup_{l=1}^{k} \operatorname{Int} B_l = \bigcup_{E \in \mathscr{E}_i} \operatorname{Int} E$. (One way to achieve this is by covering each B_l individually.) For each i, let X_i be a finite set such that

(1) $\operatorname{Int} E \cap X_i \neq \varnothing$ if $E \in \mathscr{E}_i$, (2) $X_i \subset \bigcup_{l=1}^{k} \operatorname{Int} B_l$.

Let C_1 be an n-cell with bi-collared boundary in $\bigcup_{l=1}^{k} \operatorname{Int} B_l$ such that $X_1 \subset C_1$. Applying Lemma 6 with $X = X_2 \setminus C_1$, $\mathscr{E} = \mathscr{E}_1$ and γ small, we get a $\frac{1}{2}$-homeomorphism f_1 of M onto itself such that

$$B \supset C_2 = f_1(C_1) \supset f_1(\mathring{C}_1) \supset C_1 \cup X_2,$$

$$B \supset \operatorname{supp} f_1,$$

$$f_1|(1-\gamma)C_1 = \mathrm{Id},$$

where $(1-\gamma)C_1 = C_1 \setminus \partial C_1 \times (-\gamma, 0]$.

Repeated applications of Lemma 6 give a sequence $f_1, f_2, \ldots$ of homeomorphisms of M onto itself such that for each $m \in \mathbb{N}$,

(1) f_m is a $(\frac{1}{2})^m$-homeomorphism,

(2) $B \supset f_m \circ \ldots \circ f_1(C_1) \supset f_m \circ \ldots \circ f_1(\mathring{C}_1) \supset C_1 \cup \bigcup_{i=1}^{m+1} X_i$,

(3) $\operatorname{Int} B \supset \operatorname{supp} f_m$,

(4) $f_{m+1}|f_m \circ \ldots \circ f_1((1-\gamma/2^m)C_1) = \mathrm{Id}$.

Clearly $f_m \circ \ldots \circ f_1$ converges to a map ϕ such that

$$\phi(C_1) = \lim_{m \to \infty} f_m \circ \ldots \circ f_1(C_1) = B,$$

$$\phi \text{ is a homeomorphism on } \mathring{C}_1,$$

$$\phi^{-1} \circ \phi(\partial C_1) = B \setminus \mathring{C}_1,$$

so that when ϕ is restricted to C_1 we get the required map.

REMARKS. The proofs of Propositions 1 and 4 (including its lemmas) were, in every essential aspect, taken from Fathi [3] and Morton Brown [2] respectively.

The fact that we are unable to prove Proposition 4 when $\dim M = 2$ forces us to break the proof of the following lemma into two parts.

LEMMA 7. *Let A, B be two disjoint compact sets in M^n and let $\mu \in \mathcal{M}_g^c(M)$. Then there exists a finite disjoint family of relative n-cells, say $\mathcal{K} = \{K_1, \ldots, K_r\}$, such that*

$$B \subset \bigcup_{i=1}^{r} \mathring{K}_i,$$

$$K_i \cap A = \varnothing \ (i = 1, 2, \ldots, r),$$

$$\mu(BK_i) = 0 \ (i = 1, 2, \ldots, r).$$

Furthermore, if $M^n \setminus A$ is connected then $\mathcal{K}$ can be so constructed as to contain only one element.

Proof. We have two cases to consider.

Case 1, *in which* $n = \dim M \geqslant 3$. Cover B by a finite family, say $\mathcal{E} = \{\mathring{E}_1, \ldots, \mathring{E}_t\}$, such that each E_i is a closed n-cell contained in $M \setminus A$ with $\mu(F_r E_i) = 0$ $(i = 1, 2, \ldots, t)$. By adding a few more n-cells to the collection we can assume that for every pair L_1, L_2 of distinct connected components of $\bigcup_{i=1}^{t} \operatorname{Int} E_i$ the intersection $\operatorname{Cl} L_1 \cap \operatorname{Cl} L_2$ is empty. Since $n \geqslant 3$, Proposition 4 implies that each of the components of $\bigcup_{i=1}^{t} E_i$ is a relative n-cell. If $M \setminus A$ is connected then we can add one more n-cell E to $\mathcal{E}$ to make $\operatorname{Int} E \cup \bigcup_{i=1}^{t} \operatorname{Int} E_i$ connected.

Case 2, *in which* $n = \dim M \leqslant 2$. In this case the result will be a consequence of the following facts.

First, if K is a compact connected n-manifold then K is a relative n-cell.

Secondly, for any n-manifold N $(n \leqslant 2)$ and any compact set $B \subset N$ there exists an n-dimensional submanifold L of N such that L is compact and $B \subset \mathring{L}$ (see Ahlfors and Sario [1, I.29A, I.40 and I.46]).

Applying the latter to $N = M \setminus A$, we get a compact L disjoint from A with $B \subset \mathring{L}$. If $n = 1$ then $\mu(\partial L) = 0$ since ∂L consists of a finite number of points. If

$n = 2$ then $\operatorname{Fr} L$ consists of a disjoint union of a finite number of copies of S^1 (the unit circle) and $I = [0, 1]$. Using a collar $\partial L \times [0, 1) \subset L$ we can perturb $\operatorname{Fr} L$ slightly, without moving $\partial M \cap \partial L$ (which contains $B \cap \partial L$). Hence we may assume without loss of generality that ∂L has zero μ-measure. The family $\mathscr{K} = \{K_1, \ldots, K_r\}$ of connected components of L satisfies the requirements of the lemma.

Finally, if $M \setminus A$ is connected we can assume that B is connected and hence that $B \subset \mathring{K}_i$ for some $i = 1, 2, \ldots, r$.

LEMMA 8. *Let M be a connected manifold and let $\mu \in \mathscr{M}_g^c(M)$.*

Let $\delta_1, \delta_2, \ldots, \delta_k$ be positive numbers and let $E_1, E_2, \ldots, E_k$ be a disjoint finite family of Borel sets in M such that

(1) $0 < \mu(E_i) < \infty$ $(i = 1, 2, \ldots, k)$,

and either

(2) $0 < \mu(M \setminus \bigcup E_i)$, *and* $\sum_{i=1}^{k} \delta_i < \mu(M) \leqslant \infty$,

or

(2′) $\mu(M \setminus \bigcup E_i) = 0$, *and* $\sum_{i=1}^{k} \delta_i = \mu(M)$ *(and hence* $\mu(M) < \infty$*).*

Then there exists a compactly supported $h \in \mathscr{H}^c(M)$ such that

$h_* \mu(E_i) = \delta_i$ $(i = 1, 2, \ldots, k)$,

μ and $h_* \mu$ have the same sets of measure zero.

Proof. By omitting one of the sets E_i, case (2′) can be reduced to case (2), and so we restrict ourselves to this case.

Let $E_0 = M \setminus \bigcup_{i=1}^{k} E_i$ and $E_{k+1} = M$. Since μ is (inner) regular, we can choose, for each $i = 0, 1, \ldots, k+1$, a compact set $F_i \subset E_i$ such that

$\mu(F_i) > 0$ $(0 \leqslant i \leqslant k)$,

$\mu(F_{k+1}) > \sum_{i=1}^{k} \delta_i$,

$\mu(E_i \setminus F_i) < \delta_i$ $(1 \leqslant i \leqslant k)$.

Apply Lemma 7 to $F = \bigcup_{i=0}^{k+1} F_i$ and $A = \varnothing$ (so that $M \setminus A$ is connected) to obtain a relative n-cell K with $\mu(BK) = 0$ and such that F is contained in $\mathring{K}$. By Proposition 1, K is measure conformable. Furthermore, K inherits the following properties from F:

$\mu(E_i \cap K) > 0$ $(0 \leqslant i \leqslant k)$,

$\mu(K) > \sum_{i=1}^{k} \delta_i$,

$\mu(E_i \setminus K) < \delta_i$ $(1 \leqslant i \leqslant k)$.

Define

$$\alpha_i = \delta_i - \mu(E_i \setminus K) \ (1 \leqslant i \leqslant k),$$

$$\alpha_0 = \mu(K) - \sum_{i=1}^{k} \alpha_i.$$

Then $\alpha_i > 0$ $(0 \leqslant i \leqslant k)$ and $\sum_{i=0}^{k} \alpha_i = \mu(K)$. Define $\nu \in \mathcal{M}_g^B(K)$ by the formula

$$\nu(X) = \sum_{i=0}^{k} \frac{\alpha_i}{\mu(K \cap E_i)} \mu(X \cap E_i)$$

for all $X \subset K$ Borel. We have $\nu(K) = \mu(K)$ and therefore we can find $h \in \mathscr{H}^{\partial}(M)$ with support in K such that $(h_* \mu)|K = \nu$. From the formula defining ν and the fact that $\operatorname{supp} h \subset K$ it follows that h is the required homeomorphism.

Remark. The homeomorphism h can be taken to be isotopic to the identity by an isotopy $\{h_t\}_{t \in I}$ such that $h_0 = \mathrm{Id}$, $h_1 = h$ and $\operatorname{supp} h_t \subset K$ for all $t \in I$. This follows from the remark after the proof of Proposition 1.

Definitions. Let X be a topological space and let $K \subset X$ be a compact subspace. We denote by $\mathscr{C}(X \setminus K)$ the set of connected components of $X \setminus K$. Now let $V \in \mathscr{C}(X \setminus K)$; we say that V is *bounded* if its closure is compact, and *unbounded* otherwise.

Finally, we define

$$\hat{K} = X \setminus \bigcup \{ V \in \mathscr{C}(X \setminus K) \mid V \text{ is unbounded} \}.$$

Lemma 9. *Let X be a connected, non-compact, locally connected, locally compact, Hausdorff space. Let $K \subset X$ be a compact subspace. Then $X \setminus K$ has only finitely many unbounded components and $\hat{K}$ is compact.*

Proof. If $K = \varnothing$ then the result is trivial, so assume that K is not empty. Let L be a compact neighbourhood of K. Then $\mathscr{C}(X \setminus K)$ is a disjoint open covering of the compact set $\operatorname{Fr} L$ and therefore only finitely many $V \in \mathscr{C}(X \setminus K)$ intersect (and cover) $\operatorname{Fr} L$, say $V_1, \ldots, V_q, V_{q+1}, \ldots, V_k$, where $V_1, \ldots, V_q$ are unbounded and $V_{q+1}, \ldots, V_k$ are bounded.

Now observe the following.

(1) If $V \in \mathscr{C}(X \setminus K)$ then $\operatorname{Fr} V \neq \varnothing$. Also $\operatorname{Fr} V \subset \operatorname{Fr} K$, since X is locally connected. It follows that $V \cap \mathring{L} \neq \varnothing$.

(2) If $V \in \mathscr{C}(X \setminus K)$ and $V \cap \operatorname{Fr} L = \varnothing$ then $V \subset \mathring{L}$ ($V \cap \operatorname{Fr} L = \varnothing$ implies that $V \cap \mathring{L}$ is open and closed in V so that $V \cap \mathring{L} = V$).

These observations allow us to conclude that $\hat{K} = X \Big\backslash \bigcup_{i=1}^{q} V_i$ and that $L \cup \bigcup_{i=1}^{k-q} V_{i+q}$ is a compact set containing $\hat{K}$.

DEFINITION. Let M be a manifold and let $h \in \mathscr{H}^c(M)$. We say that *h is compactly isotopic to the identity* if there exists an isotopy $\{h_t\}_{t \in I}$ such that

(1) $h_t \in \mathscr{H}^c(M)$ for every $t \in I$,

(2) $h_0 = \mathrm{Id}$ and $h_1 = h$,

(3) the support $\operatorname{supp}\{h_t\}_{t \in I} = \mathrm{Cl} \bigcup_{t \in I} \operatorname{supp} h_t$ is compact.

In what follows we shall assume that, associated with any given $h \in \mathscr{H}^c(M)$ compactly isotopic to the identity, there is a fixed isotopy $\{h_t\}_{t \in I}$ as above. We shall write $\operatorname{supp} h$ for $\operatorname{supp}\{h_t\}_{t \in I}$.

LEMMA 10. *Let M be a connected, σ-compact manifold and let A, B, K be compact subsets of M such that $A \subset \mathring{K}$ and $K = \hat{K}$. Let $\mu_1, \mu_2 \in \mathscr{M}_g^c(M)$ agree on ends and suppose further that $\mu_1|K = \mu_2|K$ and $\mu_1(V) = \mu_2(V)$ for every $V \in \mathscr{C}(M \setminus K)$. Then there exists an $h \in \mathscr{H}^c(M)$ compactly isotopic to the identity and a compact set L such that*

(1) $\operatorname{supp} h \cap A = \varnothing$,

(2) $\mathring{L} \supset K \cup B$ *and* $L = \hat{L}$,

(3) $h_* \mu_1 | L = \mu_2 | L$,

(4) $(h_* \mu_1)(W) = \mu_2(W)$ *for all* $W \in \mathscr{C}(M \setminus L)$.

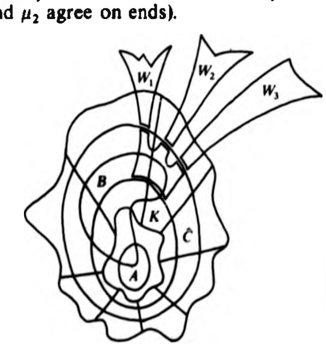
Proof. Let C be a compact set in M such that $K \cup B \subset \mathring{C}$. By applying Lemma 7 to $\hat{C} \setminus \mathring{K}$ and A, with $\mu = \mu_1 + \mu_2$, we can find a disjoint finite family of compact measure conformable sets $L_1, \dots, L_p$ such that

$$\mu_1(BL_i) = \mu_2(BL_i) = 0 \ (i = 1, 2, \dots, p),$$

$$L_i \cap A = \varnothing \ (i = 1, 2, \dots, p),$$

$$\hat{C} \subset \mathring{K} \cup \bigcup_{i=1}^{p} \mathring{L}_i.$$

Let $V \in \mathscr{C}(M \setminus K)$ and let $W_1, \dots, W_q$ be the components of $V \setminus \hat{C}$. Observe that, for every $j = 1, 2, \dots, q$, $\mu_1(W_j) = \infty$ if and only if $\mu_2(W_j) = \infty$ (this follows from the assumption that μ_1 and μ_2 agree on ends).



Each "slice of cake" in the diagram represents some L_i.

We have a decomposition of V into disjoint sets

$$L_i \cap V \setminus \bigcup_{j=1}^{q} W_j \ (i = 1, 2, \ldots, p),$$

$$L_i \cap W_j \ (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q),$$

$$W_j \setminus \bigcup_{i=1}^{p} L_i \ (j = 1, 2, \ldots, q).$$

Lemma 8 tells us that we can construct a homeomorphism compactly isotopic in V to the identity, with certain properties. It follows that each set in the decomposition may be assumed, without loss of generality, to have the same total μ_1 and μ_2-measure. Clearly we can do this simultaneously for each $V \in \mathscr{C}(M \setminus K)$.

We now have $\mu_1(L_i) = \mu_2(L_i)$ for $1 \leqslant i \leqslant p$. Using the fact that each L_i is measure conformable, we may assume that μ_1 and μ_2 are identical measures in $\bigcup_{i=1}^{p} L_i$, hence in all of $K \cup \bigcup_{i=1}^{p} L_i$. Letting $L = \bar{C}$ we have $\mu_1|L = \mu_2|L$ and $\mu_1(W) = \mu_2(W)$ for each $W \in \mathscr{C}(M \setminus L)$. This concludes the proof.

Proof of the main theorem

Let $M = \bigcup_{i=1}^{\infty} C_i$, where C_i is compact and $C_i \subset \mathring{C}_{i+1}$ $(i = 1, 2, \ldots)$. Applying Lemma 10 to $A_0 = \varnothing$, $B_0 = C_1$, $K_0 = \varnothing$ we get $h_1 \in \mathscr{H}^c(M)$ compactly isotopic to the identity and a compact set L_1 such that

$$\mathring{L}_1 \supset C_1 \text{ and } L_1 = \hat{L}_1,$$

$$h_{1*}\mu_1|L_1 = \mu_2|L_1,$$

$$(h_{1*}\mu_1)(W) = \mu_2(W) \text{ for each } W \in \mathscr{C}(M \setminus L_1).$$

Now let $A_1 = C_1$, $B_1 = C_2 \cup \operatorname{supp} h_1$, $K_1 = L_1$ and, using Lemma 10, choose $h_2 \in \mathscr{H}^c(M)$ compactly isotopic to the identity and L_2 compact such that

$$\operatorname{supp} h_2 \cap A_1 = \varnothing,$$

$$\mathring{L}_2 \supset L_1 \cup C_2 \cup \operatorname{supp} h_1 \text{ and } L_2 = \hat{L}_2,$$

$$h_{1*}\mu_1|L_2 = h_{2*}\mu_2|L_2,$$

$$(h_{1*}\mu_1)(W) = (h_{2*}\mu_2)(W) \text{ for all } W \in \mathscr{C}(M \setminus L_2).$$

Then we can define, by induction, a sequence $L_0 = \varnothing, L_1, L_2, \ldots$ of compact sets, and a sequence $h_0 = \mathrm{Id}, h_1, h_2, \ldots$ of homeomorphisms (compactly isotopic to the identity) such that

(1) $L_i = \hat{L}_i$ $(i = 0, 1, 2, \ldots)$,

(2) if $n \geqslant 1$ and $A_n = L_{n-1} \cup C_n \cup \operatorname{supp} h_{n-1}$, then $\mathring{L}_n \supset A_n$ and $\operatorname{supp} h_{n+1} \cap A_n = \varnothing$,

(3) $(h_{n+1} \circ h_{n-1} \circ \ldots \circ h_1)_* \mu_1 | L_{n+1} = (h_n \circ h_{n-2} \circ \ldots \circ h_0)_* \mu_2 | L_{n+1}$ for n even, and $(h_n \circ h_{n-2} \circ \ldots \circ h_1)_* \mu_1 | L_{n+1} = (h_{n+1} \circ h_{n-1} \circ \ldots \circ h_0)_* \mu_2 | L_{n+1}$ for n odd.

Clearly

$$h_{\text{even}} = \lim_{n \to \infty} h_{2n} \circ h_{2n-2} \circ \ldots \circ h_2 \circ h_0$$

$$h_{\text{odd}} = \lim_{n \to \infty} h_{2n+1} \circ h_{2n-1} \circ \ldots \circ h_1$$

exist in $\mathscr{H}^c(M)$, $h_{\text{odd}*}\mu_1 = h_{\text{even}*}\mu_2$ and h_{even}, h_{odd} are isotopic to the identity.

Remarks

To say that a connected manifold M is paracompact is equivalent to saying that (1) it is σ-compact, or (2) it has a countable basis, or (3) it is metrizable or (4) it is embeddable in $\mathbb{R}^N$ for some integer N. Paracompactness is not equivalent to separability (that is, the existence of a countable dense subset).

One may ask whether the main theorem is true for non-paracompact manifolds. To avoid any possible indecision about the meaning of an "end", one can restrict the question to finite measures.

Alternatively one can use the definition of an end given in this paper. In that case one can construct examples of good regular measures which are σ-finite and of other measures which are not σ-finite, having the same behaviour on ends. Clearly such examples are not homeomorphic. It seems more reasonable to restrict the question to σ-finite good measures μ, which includes the case of a finite measure, mentioned in the preceding paragraph. In that case the regularity of the measure implies that M has an open submanifold N which is paracompact, and such that $\mu(M \setminus N) = 0$ (in particular N is dense in M, so that M is separable). In this case our theorem can be applied to N. Unfortunately we are unable to show that this implies the truth of the theorem for M itself. We leave this as an open question.

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